photopills.com

Simple Traces

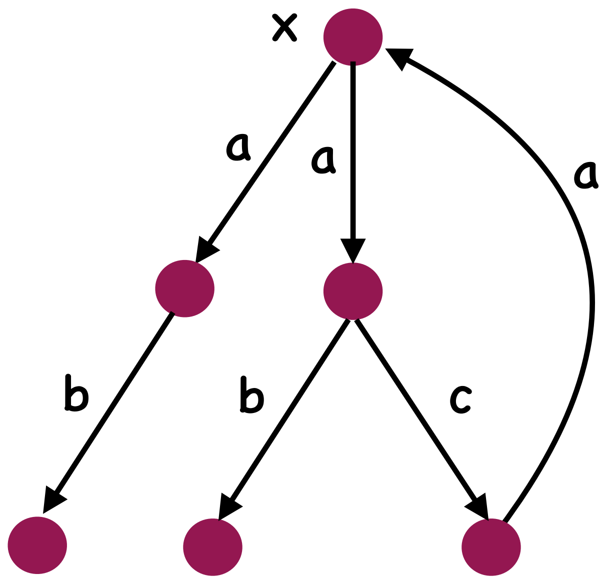
Ana Sokolova

PARIS LODRON UNIVERSITY SALZBURG

OPCT 2023, Bertinoro

Labelled Transition Systems

a basic model for step-by-step execution of a program



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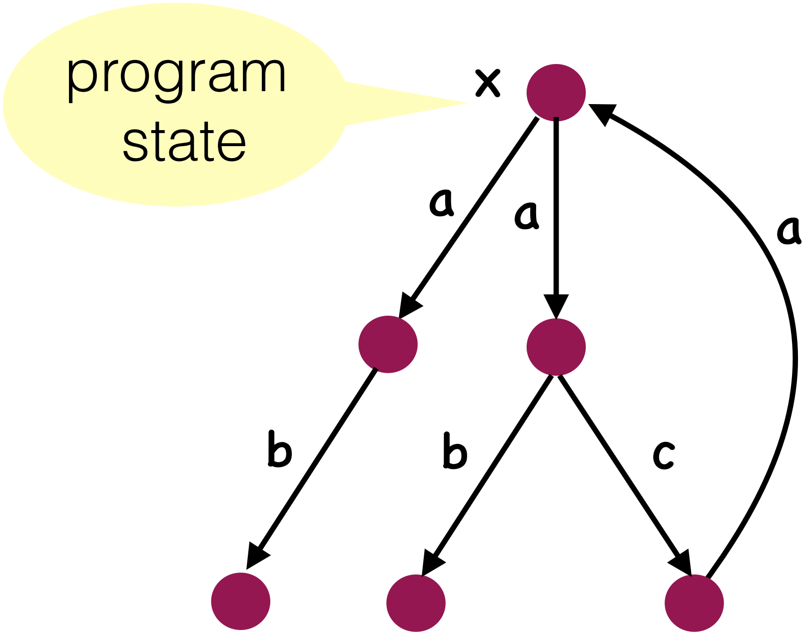
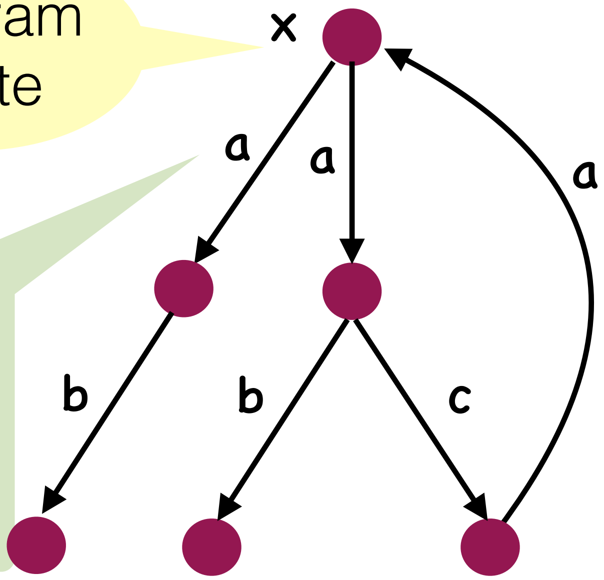

program state

Labelled Transition Systems

a basic model for step-by-step execution of a program



program state

label = instruction, input, interaction with the environment,…

Labelled Transition Systems

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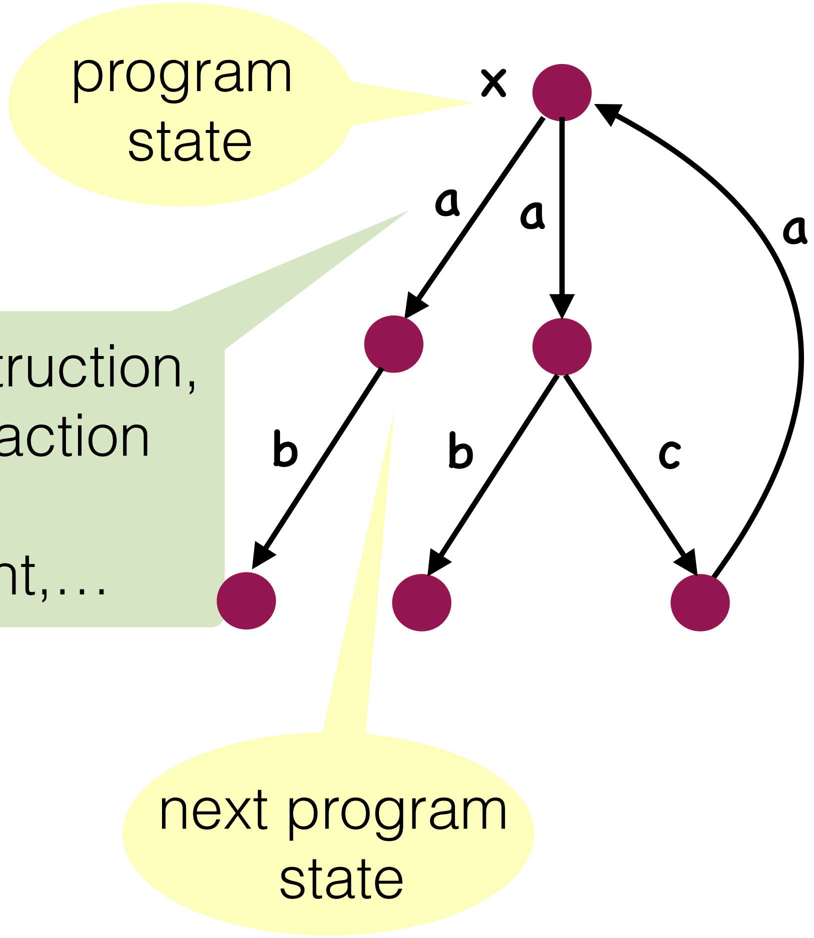


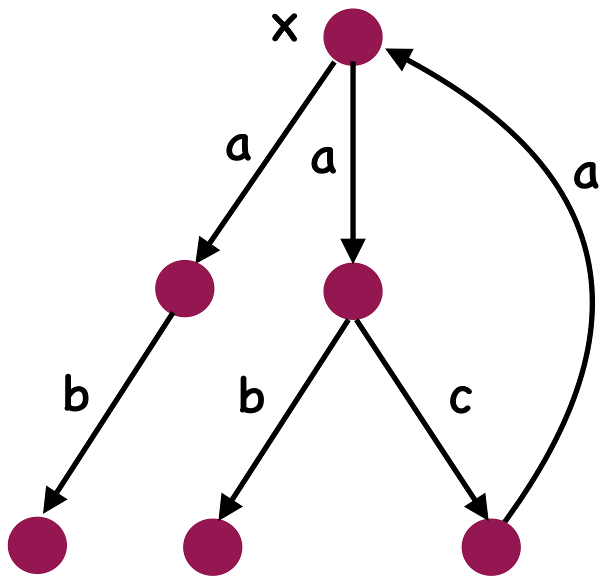
program state

label = instruction, input, interaction with the environment,…

next program state

Labelled Transition Systems

a basic model for step-by-step execution of a program



$$X \to (\mathcal{P}X)^A$$

Labelled Transition Systems

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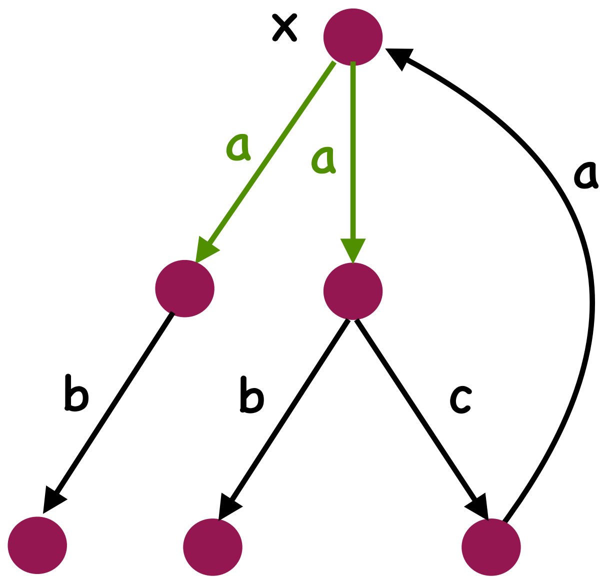
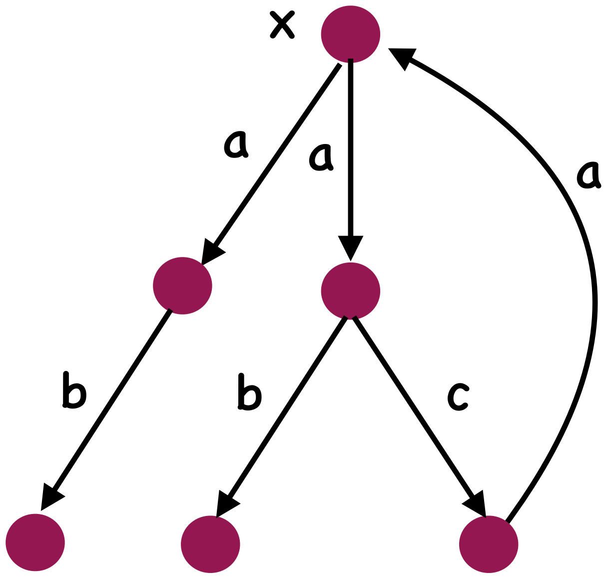


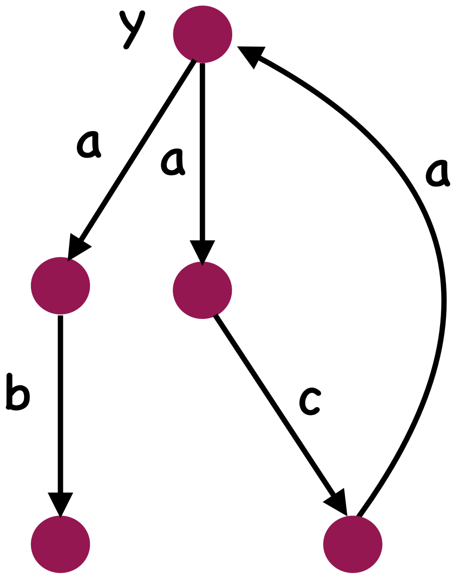
$$X \to (\mathcal{P}X)^A$$

nondeterminism (concurrency)

Equivalence of Labelled Transition Systems



$\overset{?}{\equiv}$



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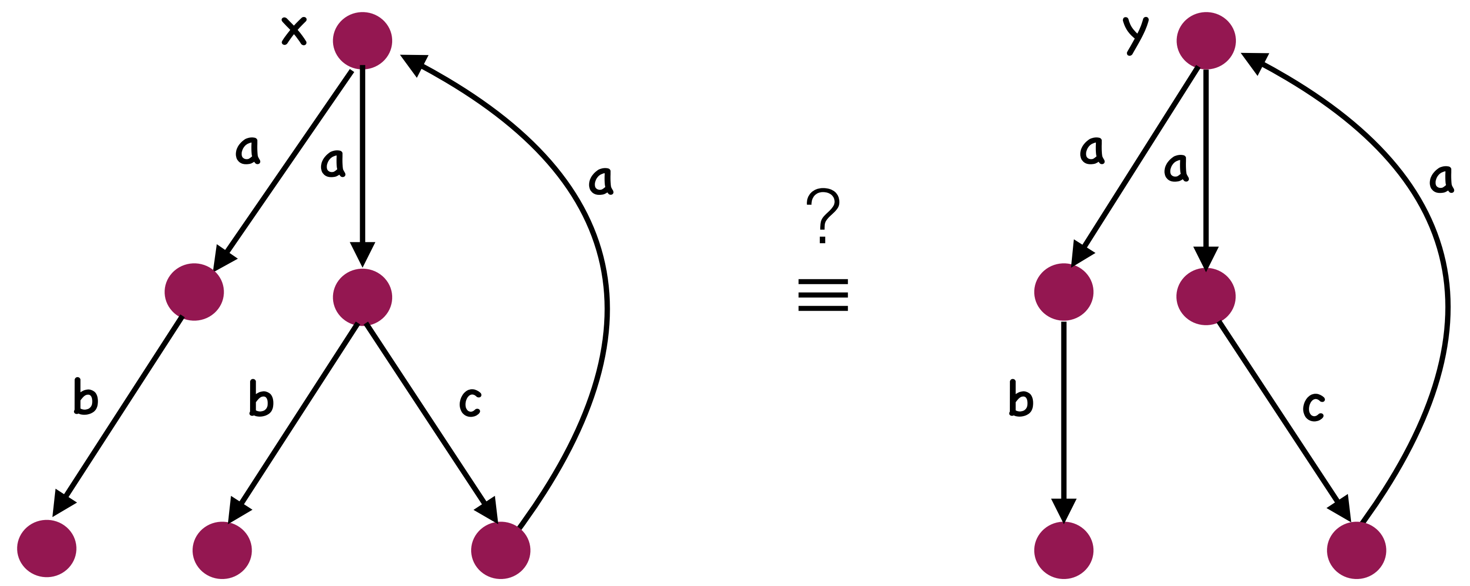
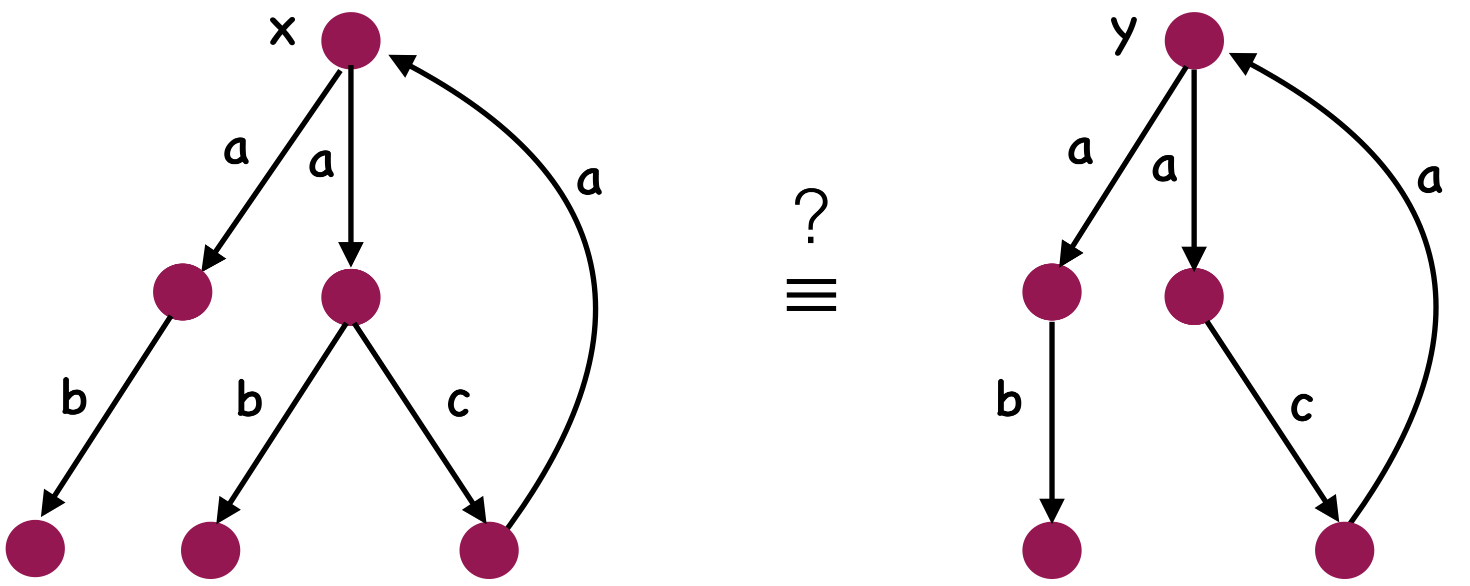


trace equivalent

perform the same traces
(=sequences of labels)

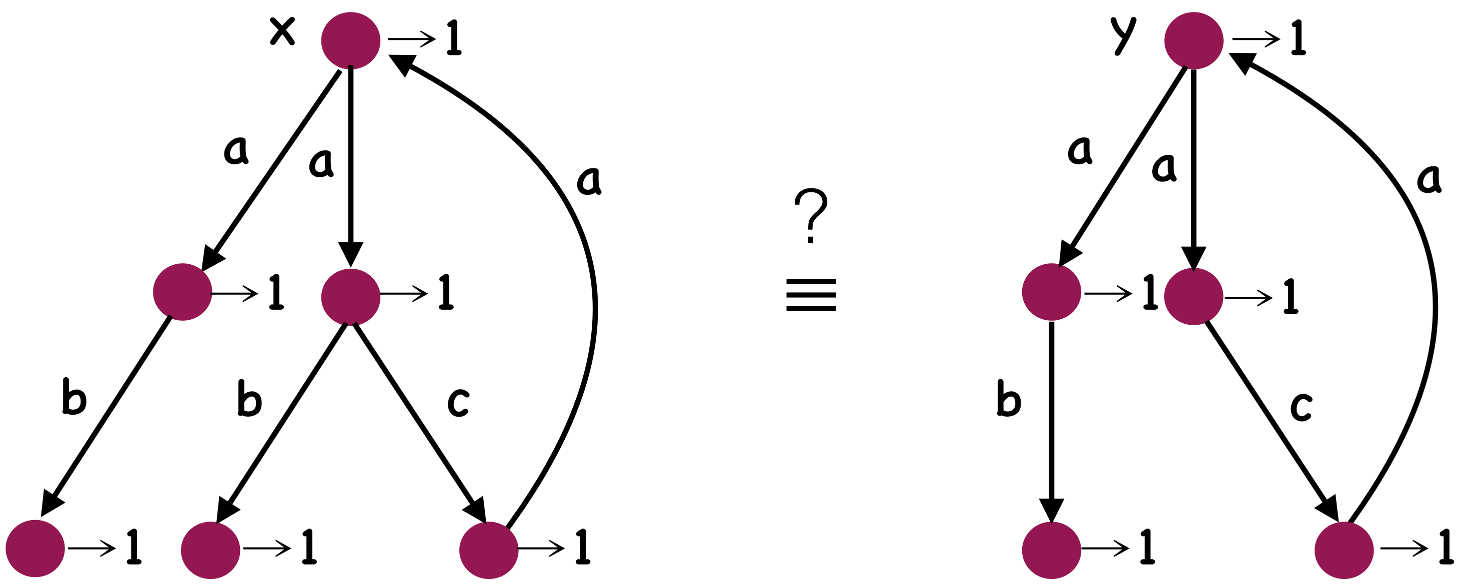
Equivalence of Labelled Transition Systems



trace equivalent

language equivalent
(LTS = automaton with all states final)

Equivalence of Labelled Transition Systems



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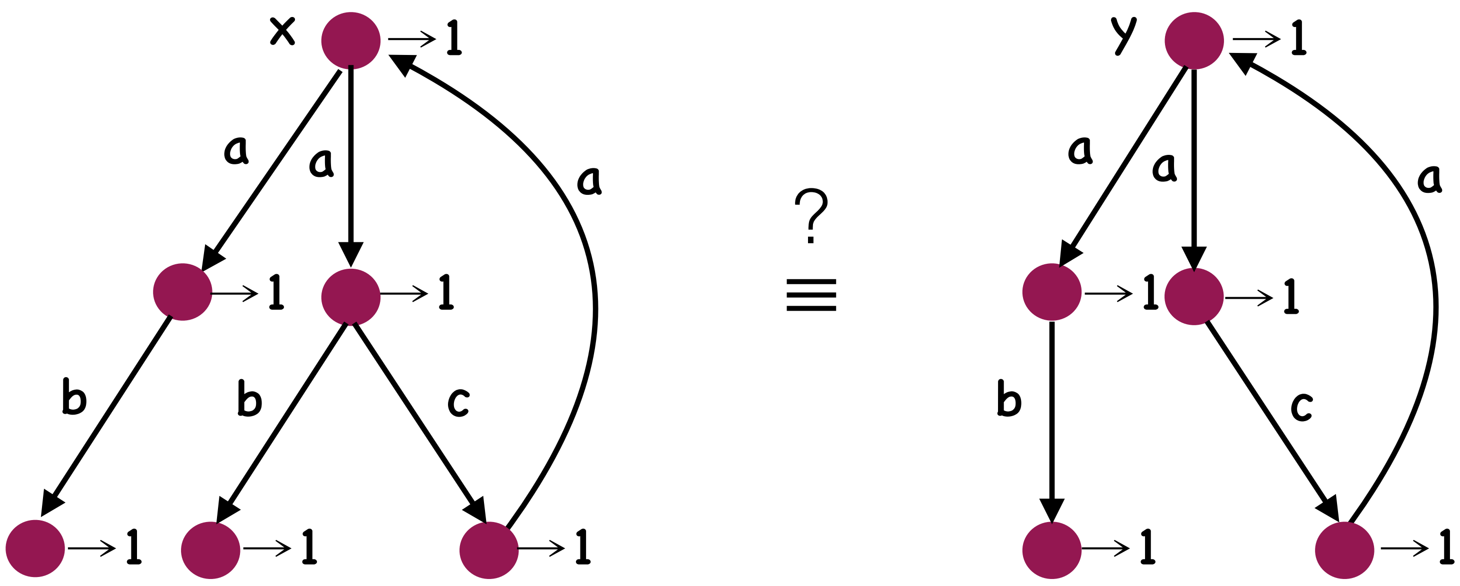
Equivalence of Labelled Transition Systems

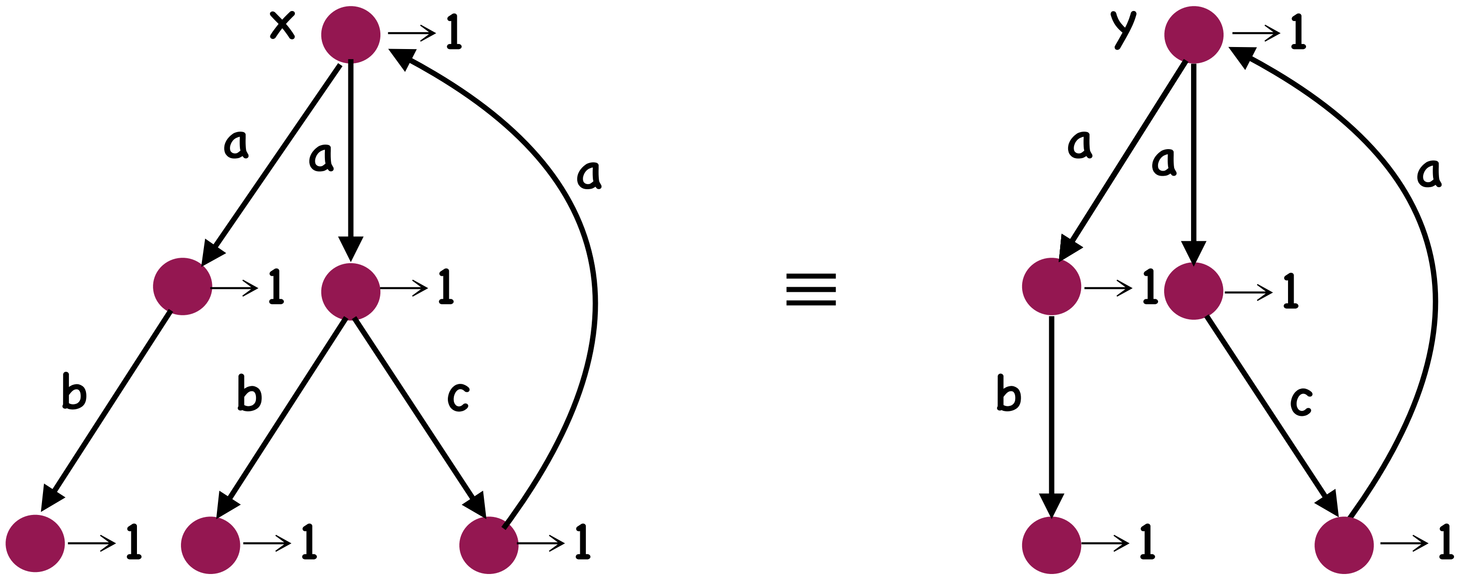


trace equivalent

language equivalent
(LTS = automaton with all states final)

$$X \to \{0,1\} \times (\mathcal{P}X)^A$$

Equivalence of Labelled Transition Systems



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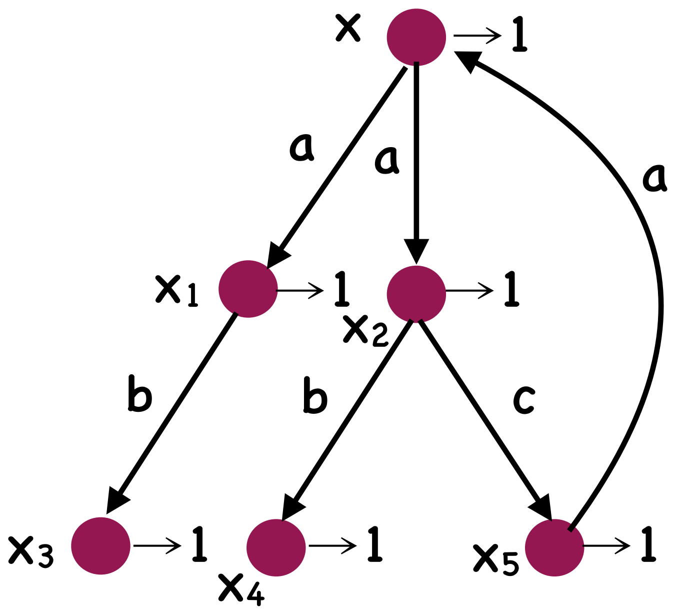

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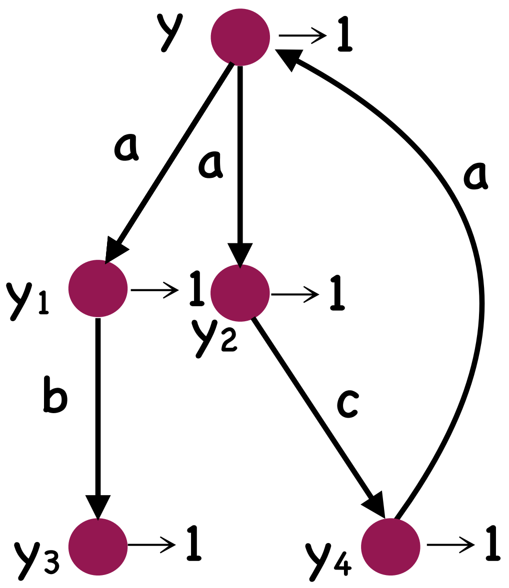

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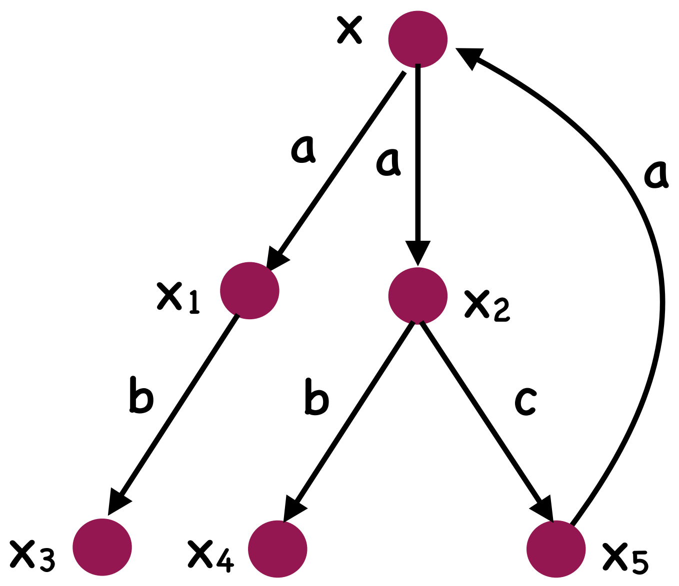

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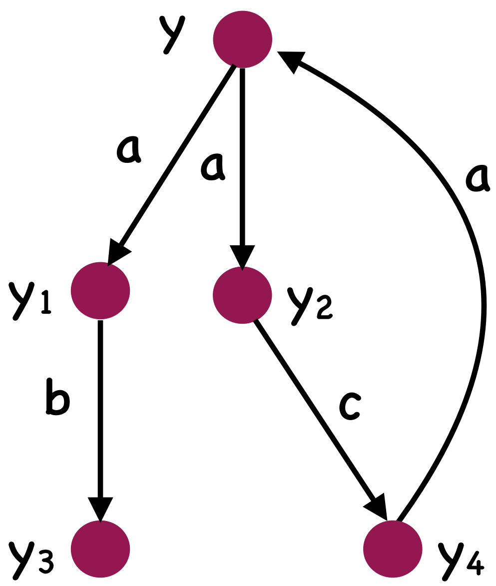

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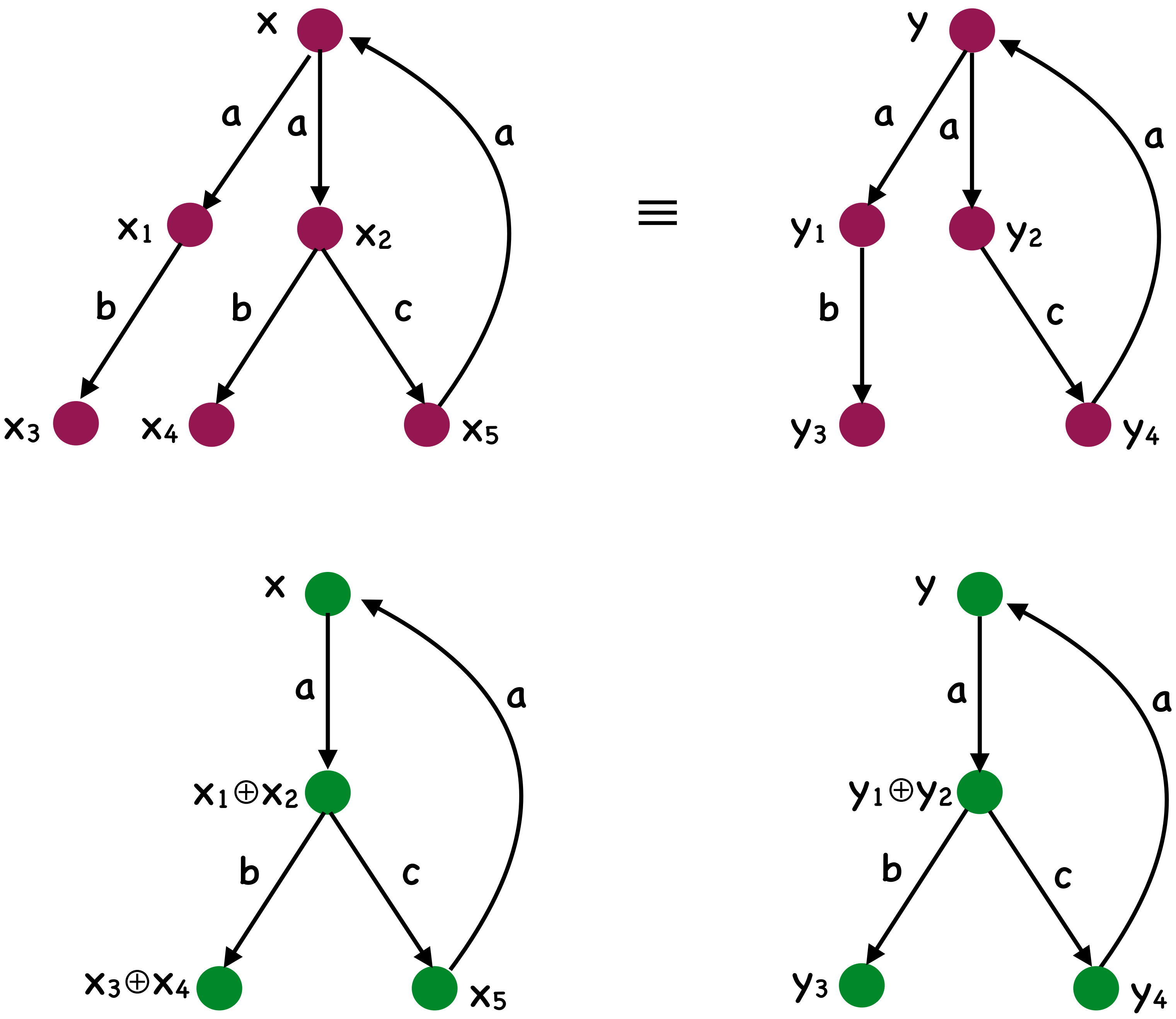

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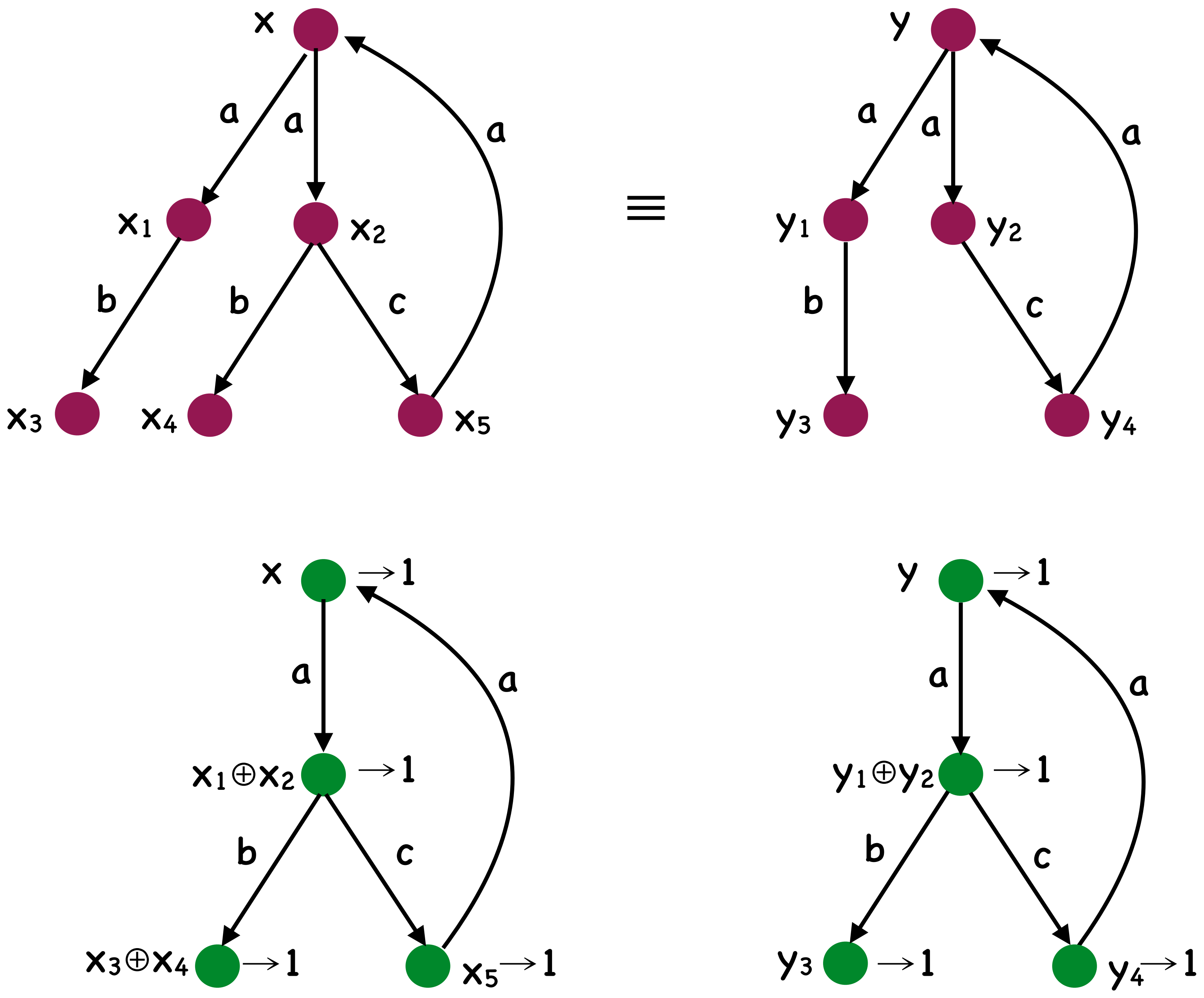
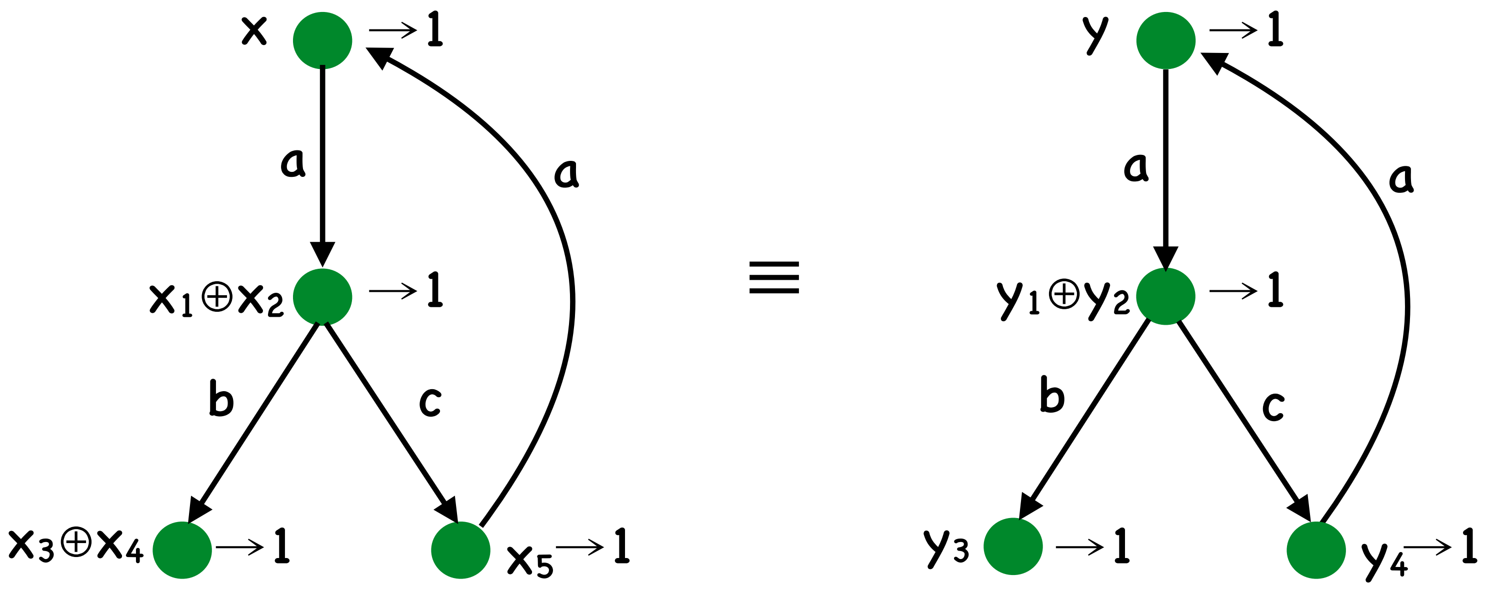


JSLB
(sup)

⊕	0	1
0	0	1
1	1	1

Equivalence of Labelled Transition Systems



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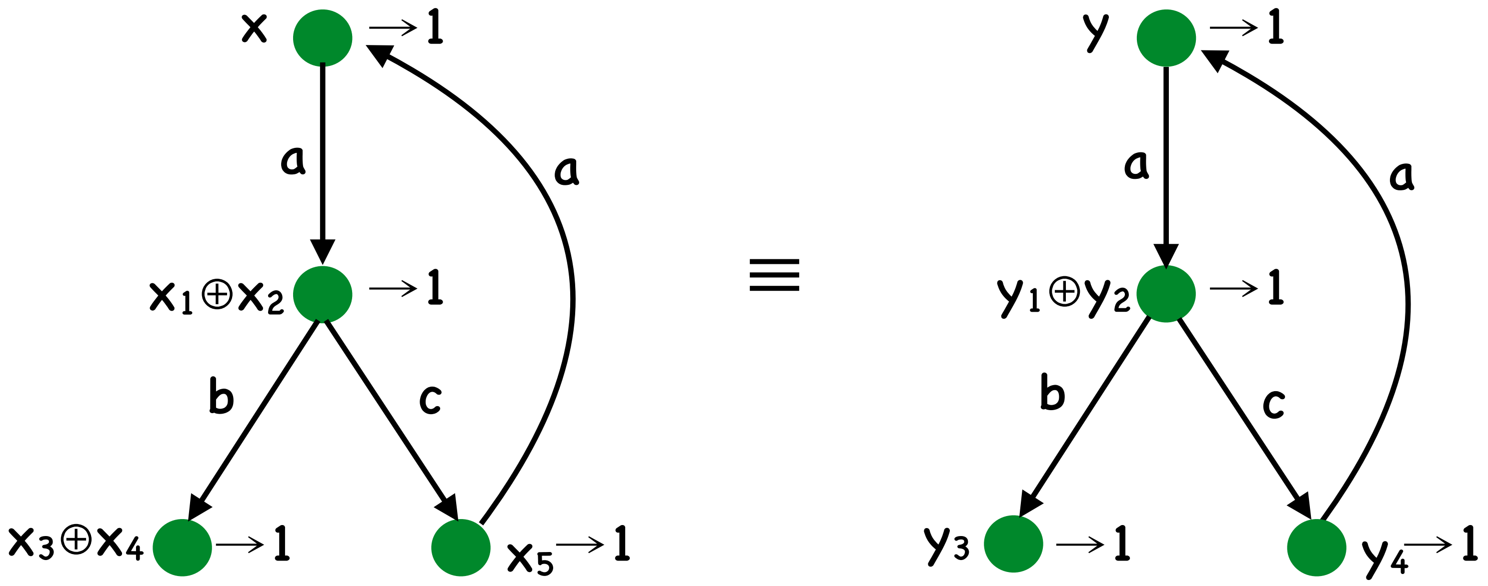

$$x \xrightarrow{acaab} x_3 \oplus x_4 \to 1 \oplus 1 = 1$$

JSLB
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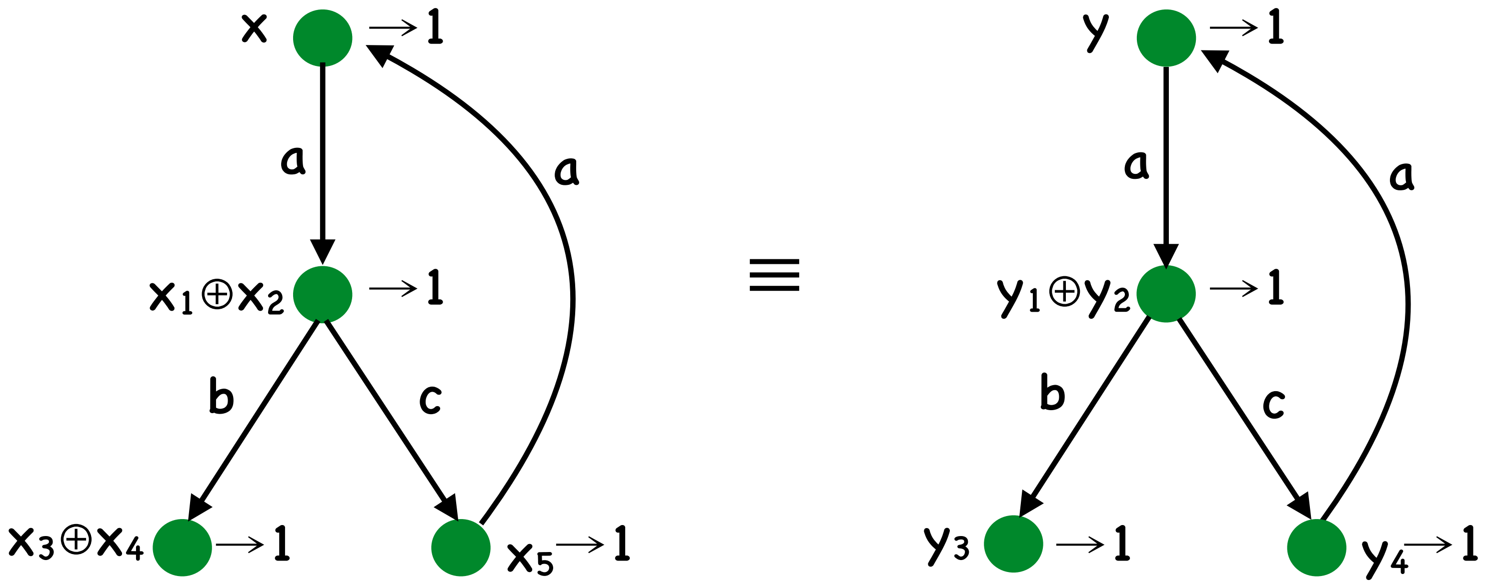
Equivalence of Labelled Transition Systems



$x \xrightarrow{acaab} x_3 \oplus x_4 \to 1 \oplus 1 = 1$

$y \xrightarrow{acaab} y_3 \to 1$

JSLB
(sup)

$\oplus$	0	1
0	0	1
1	1	1

Labelled Transition Systems

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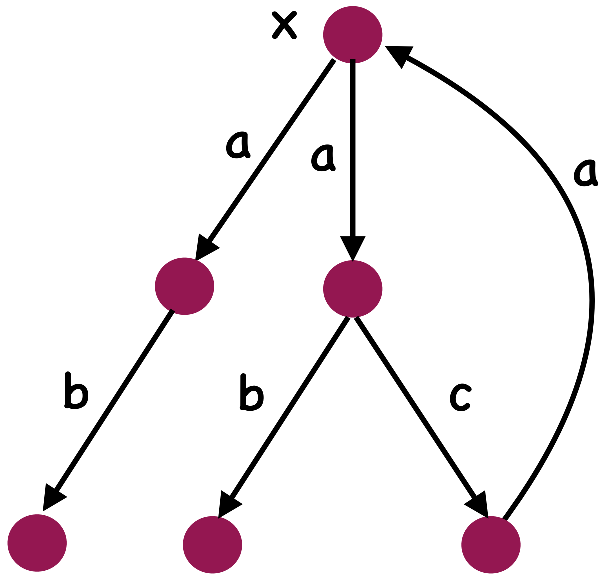


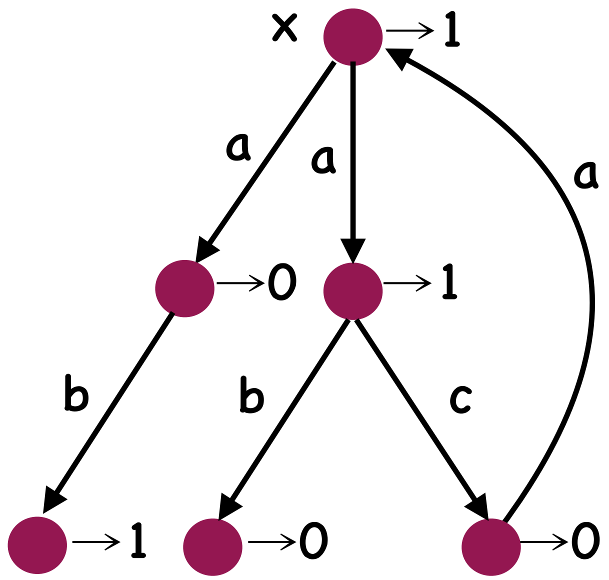
$$X \to (\mathcal{P}X)^A$$

$$X \to (\text{Free-JSLB } X)^A$$

Free JSLB {1}

⊕	0	1
0	0	1
1	1	1

Nondeterministic Automata



Free
JSLB {1}

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1	1	1

$$X \to \{0,1\} \times (\mathcal{P}X)^A$$

$$X \to \{0,1\} \times (\text{Free-JSLB } X)^A$$

The Algebraic Theory of (Join) Semilattices with Bottom

$$s, t \ ::= \ \star, \ s \oplus t, \ x \in X$$

$$\begin{array}{ccc}
(x \oplus y) \oplus z & \overset{(A)}{=} & x \oplus (y \oplus z) \\
x \oplus y & \overset{(C)}{=} & y \oplus x \\
x \oplus x & \overset{(I)}{=} & x \\
x \oplus \star & \overset{(B)}{=} & x
\end{array}$$

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this theory is a presentation for the powerset monad

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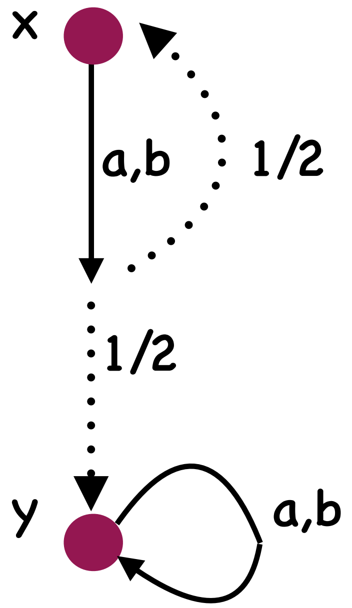
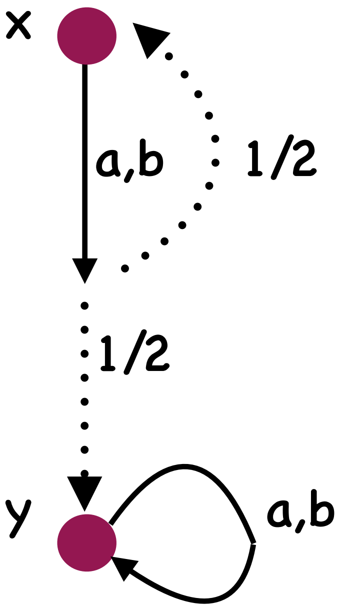
finite

Probabilistic Transition Systems



$$X \to (\mathcal{D}X)^A$$

Probabilistic Transition Systems



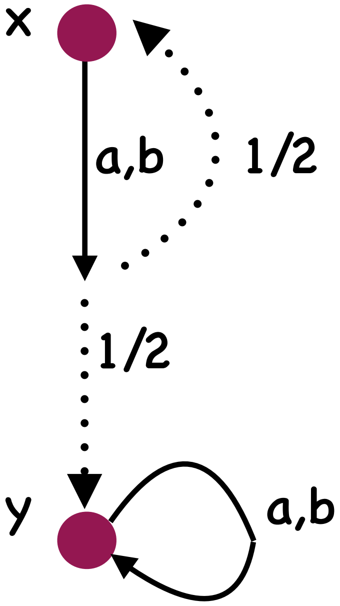
Free CA {1}

$\{1\}, +_p$

$X \to (\mathcal{D}X)^A$

$X \to (\text{Free-CA } X)^A$

Probabilistic Transition Systems



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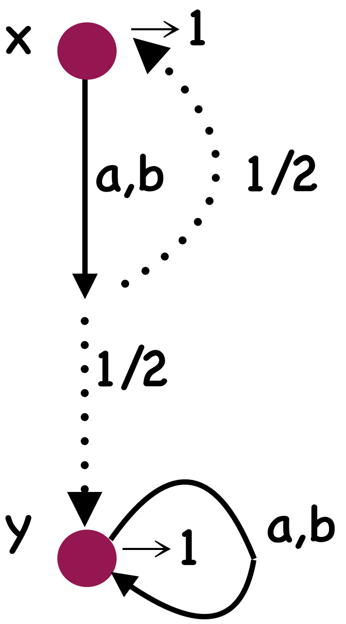
Probabilistic Transition Systems



trivial

Free CA {1}

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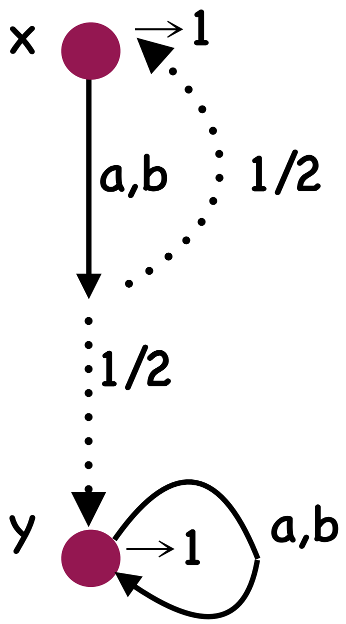
$X \to (\mathcal{D}X)^A$

$X \to (\text{Free-CA } X)^A$

Probabilistic Transition Systems

$$\begin{tikzcd}
x \arrow[d, "a"] \\
x +_{1/2} y \rightarrow 1 +_{1/2} 1 = 1 \arrow[d, "a"] \\
(x +_{1/2} y) +_{1/2} y = x +_{1/4} y \rightarrow 1 +_{1/4} 1 = 1
\end{tikzcd}$$



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Probabilistic Transition Systems

$$\begin{tikzcd}
x \arrow[d, "a"] \arrow[dd, bend right=60, "aa"'] \\
x +_{1/2} y \to 1 +_{1/2} 1 = 1 \arrow[d, "a"] \\
(x +_{1/2} y) +_{1/2} y = x +_{1/4} y \to 1 +_{1/4} 1 = 1
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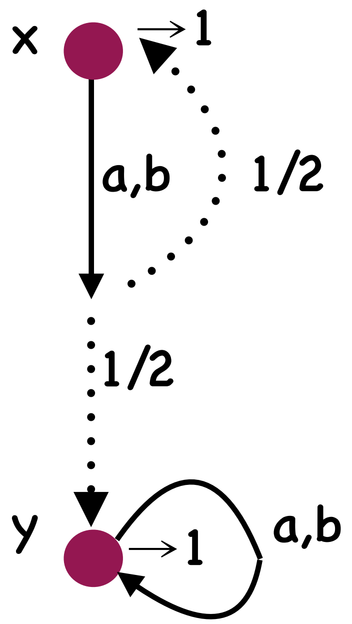

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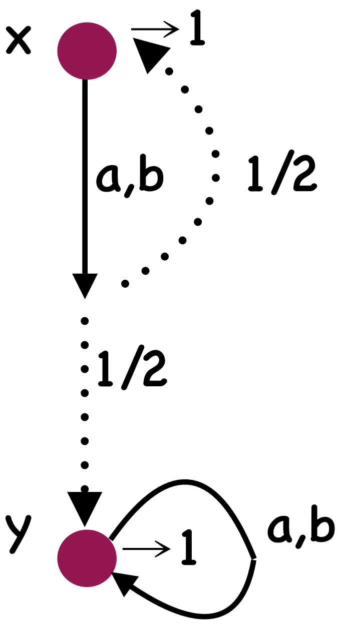
Probabilistic Transition Systems



subdistributions

Free PCA {1}

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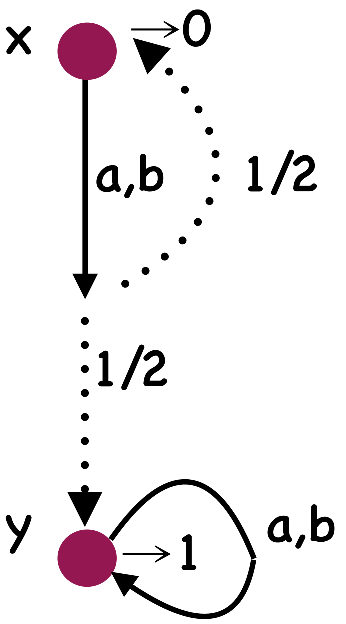
Probabilistic Automata



Free PCA {1}

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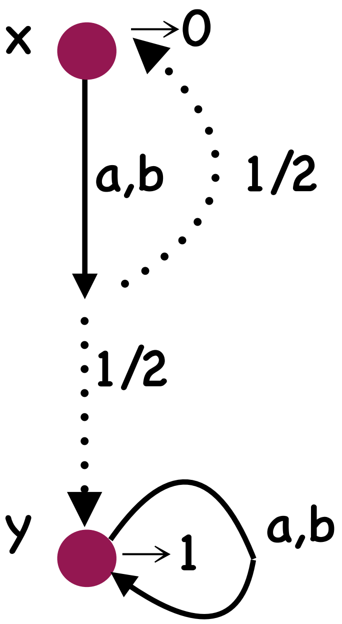
Probabilistic Automata

$$\begin{tikzcd}
x \arrow[d, "a"] \\
x +_{1/2} y \to 0 +_{1/2} 1 = 1/2 \arrow[d, "a"] \\
(x +_{1/2} y) +_{1/2} y = x +_{1/4} y \to 0 +_{1/4} 1 = 3/4
\end{tikzcd}$$



Free PCA {1}

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$X \to [0,1] \times (\mathcal{D}X)^A$

$X \to [0,1] \times (\text{Free-PCA } X)^A$

The Algebraic Theory of Convex Algebras

$$s, t \ ::= \ s +_p t, \ x \in X \quad \text{for all } p \in [0,1]$$

$$\begin{array}{ccc} (x +_q y) +_p z & \overset{(A_p)}{=} & x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z) \\ x +_p y & \overset{(C_p)}{=} & y +_{1-p} x \\ x +_p x & \overset{(I_p)}{=} & x \end{array}$$

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this theory is a presentation for the distribution monad

The Algebraic Theory of Convex Algebras

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x +_p x &\overset{(I_p)}{=} x
\end{aligned}$$

The set of terms quotiented by these axioms is isomorphic to $\mathcal{D}X$

this theory is a presentation for the distribution monad

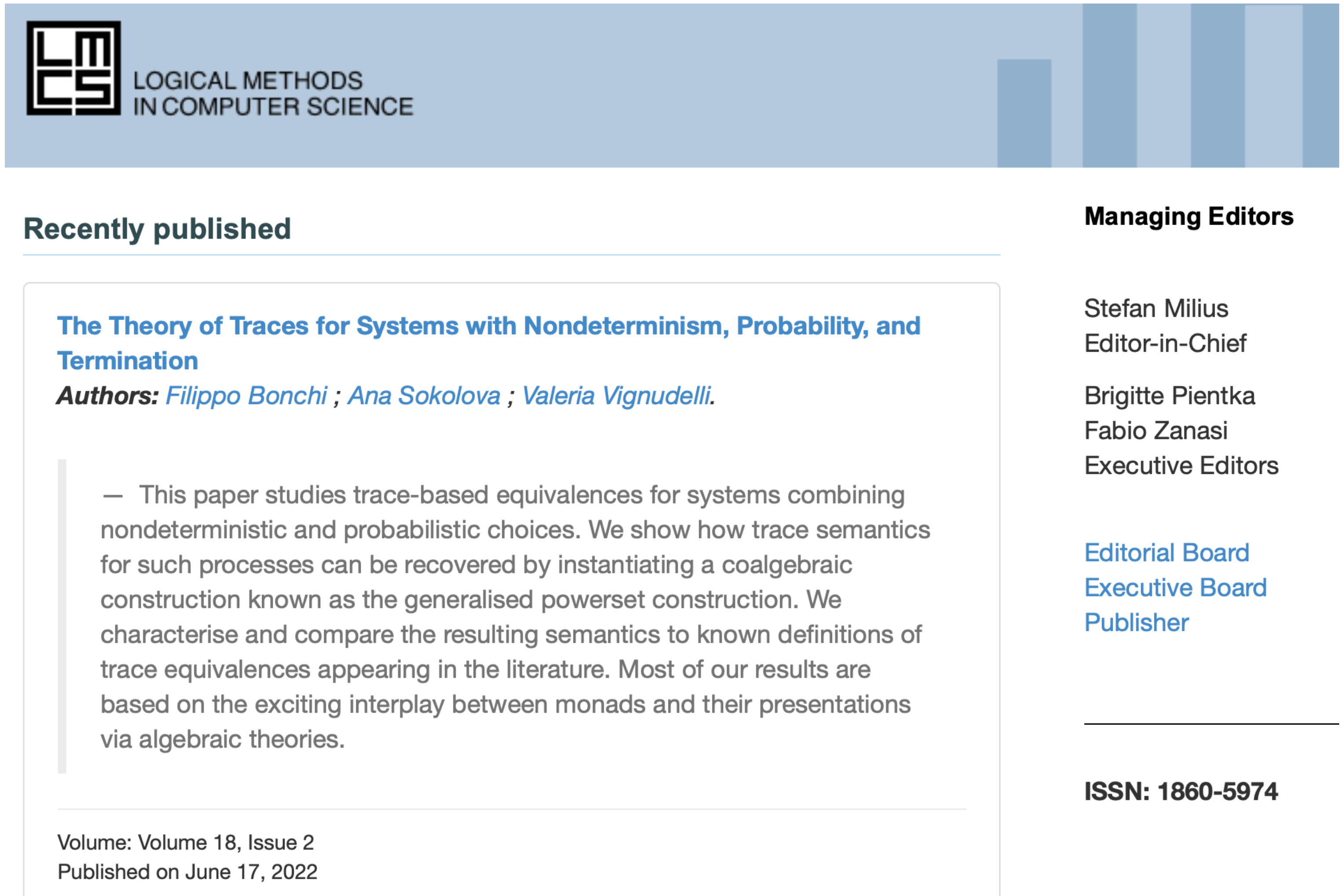
finitely supported

Nondeterministic Probabilistic Transition Systems

Bonchi, S., Vignudelli LICS 2019

LICS Distinguished Paper

Journal: LMCS 2022



LOGICAL METHODS
IN COMPUTER SCIENCE

Recently published

The Theory of Traces for Systems with Nondeterminism, Probability, and Termination

Authors: *Filippo Bonchi ; Ana Sokolova ; Valeria Vignudelli.*

> — This paper studies trace-based equivalences for systems combining nondeterministic and probabilistic choices. We show how trace semantics for such processes can be recovered by instantiating a coalgebraic construction known as the generalised powerset construction. We characterise and compare the resulting semantics to known definitions of trace equivalences appearing in the literature. Most of our results are based on the exciting interplay between monads and their presentations via algebraic theories.

Volume: Volume 18, Issue 2
Published on June 17, 2022

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A proof method for trace equivalence

nondeterministic
labelled
transition system

A proof method for trace equivalence

nondeterministic
labelled
transition system

determinisation

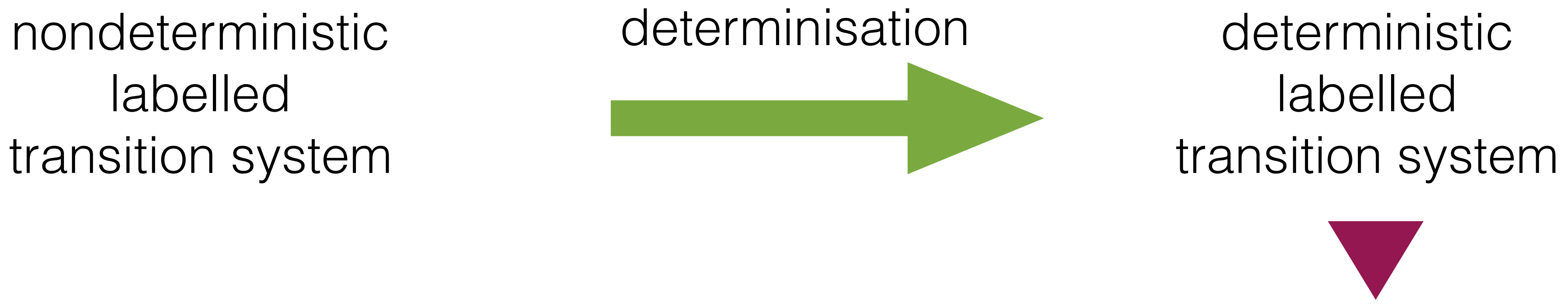
A proof method for trace equivalence

nondeterministic
labelled
transition system

determinisation →

deterministic
labelled
transition system

A proof method for trace equivalence



nondeterministic
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transition system

determinisation

deterministic
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A proof method for trace equivalence

nondeterministic
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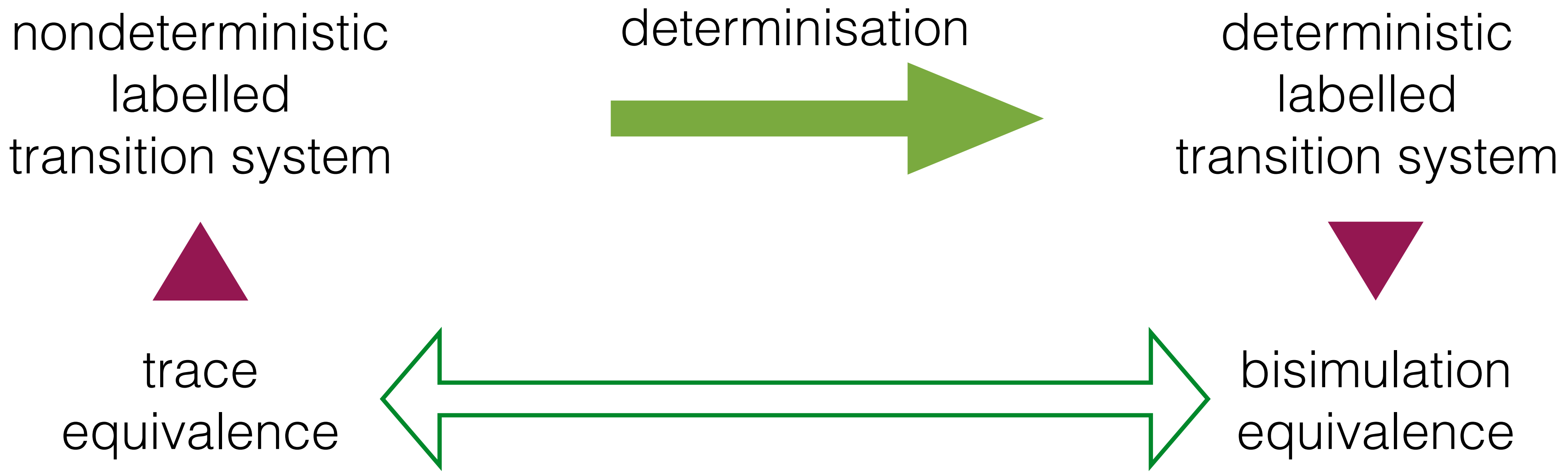
determinisation

deterministic
labelled
transition system

bisimulation
equivalence

A proof method for trace equivalence



nondeterministic labelled transition system

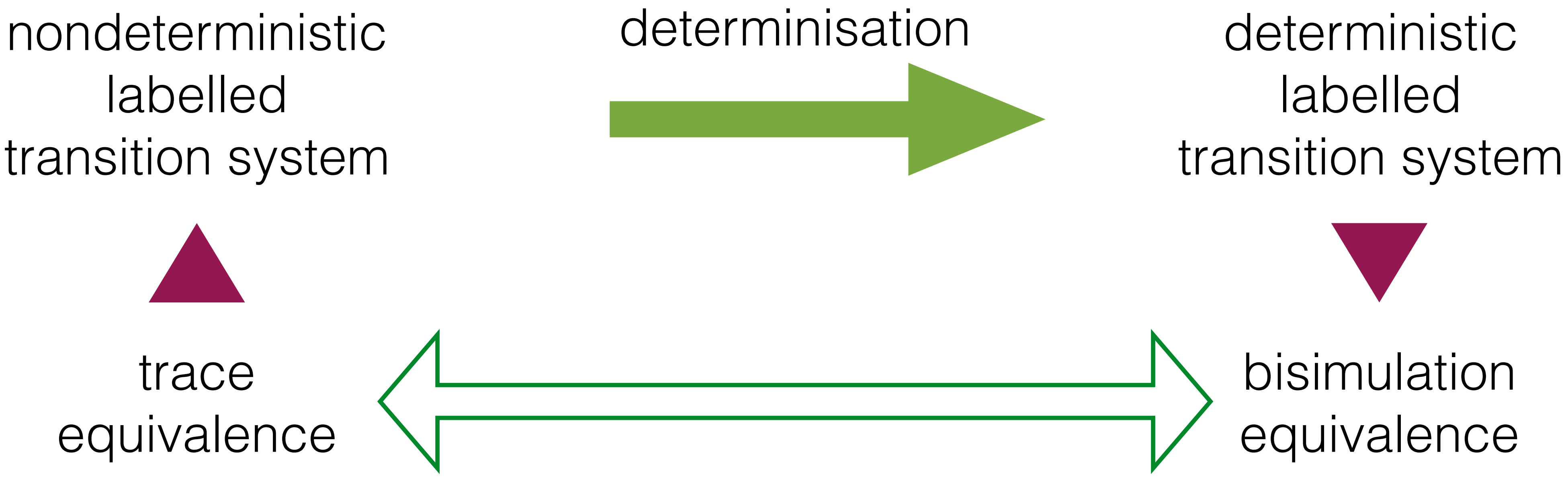
determinisation

deterministic labelled transition system

trace equivalence

bisimulation equivalence

A proof method for trace equivalence



nondeterministic
labelled
transition system

determinisation

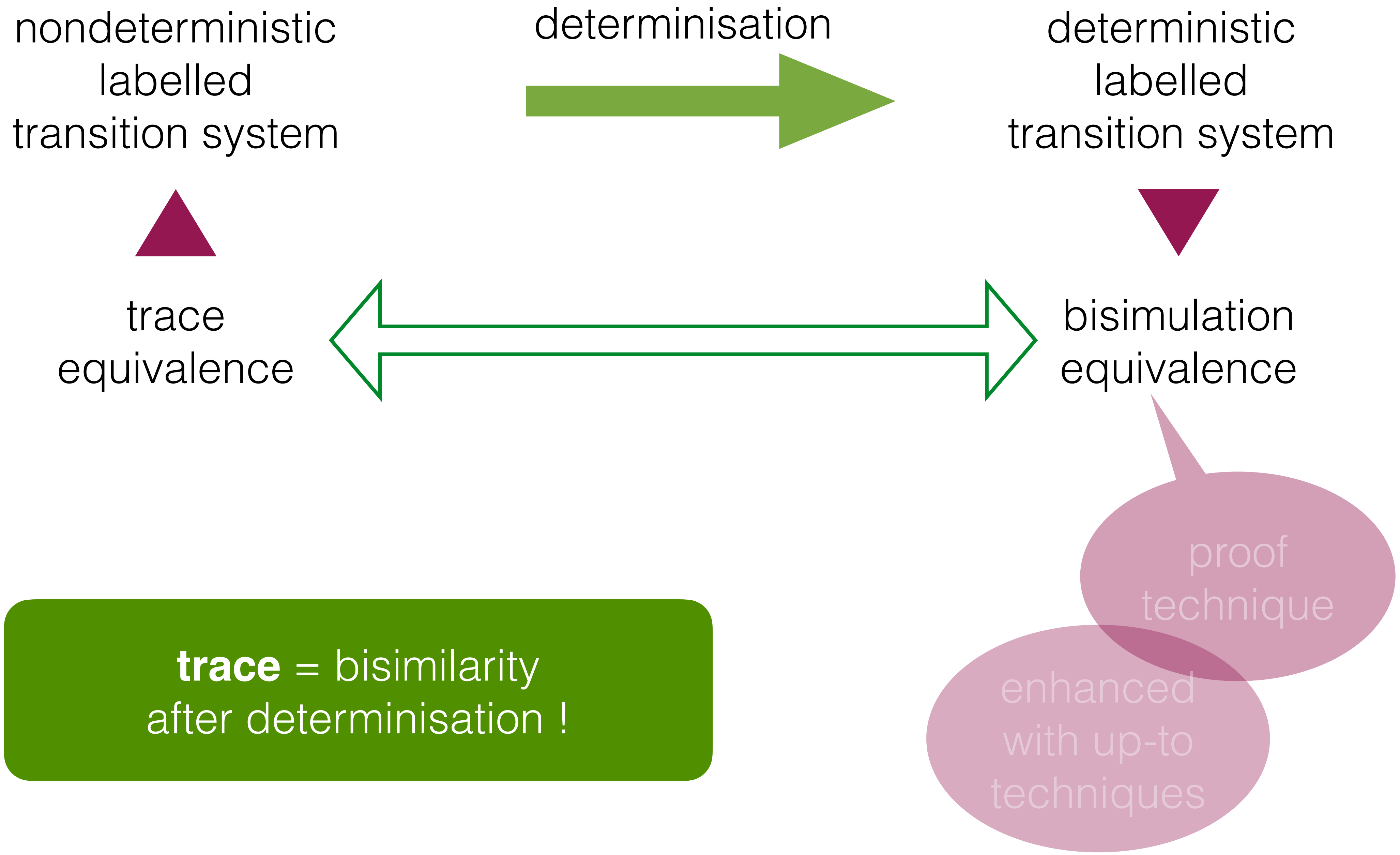
deterministic
labelled
transition system

trace
equivalence

bisimulation
equivalence

trace = bisimilarity
after determinisation !

A proof method for trace equivalence



nondeterministic
labelled
transition system

determinisation

deterministic
labelled
transition system

trace
equivalence

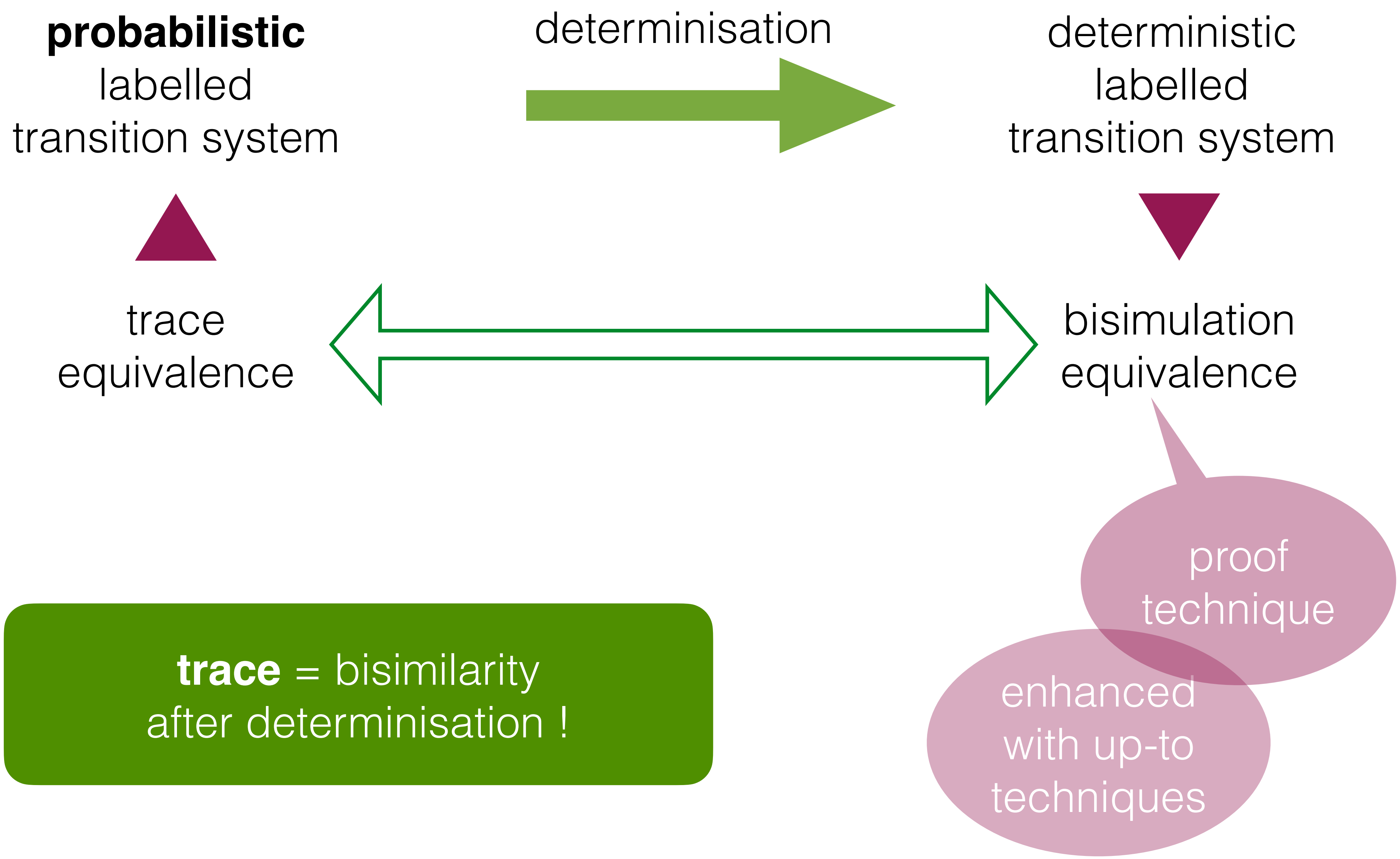
bisimulation
equivalence

proof
technique

enhanced
with up-to
techniques

trace = bisimilarity
after determinisation !

A proof method for trace equivalence



probabilistic
labelled
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determinisation

deterministic
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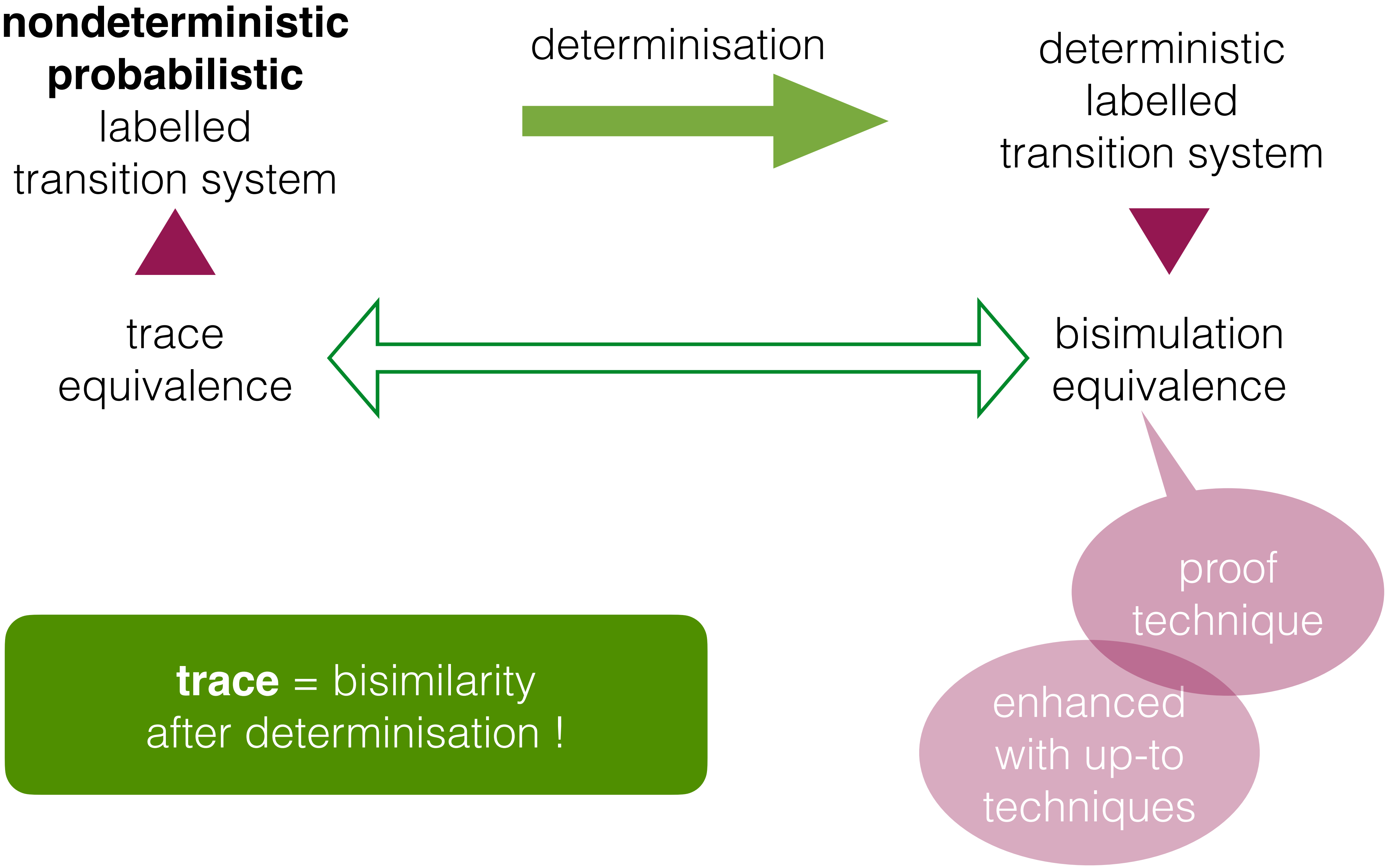

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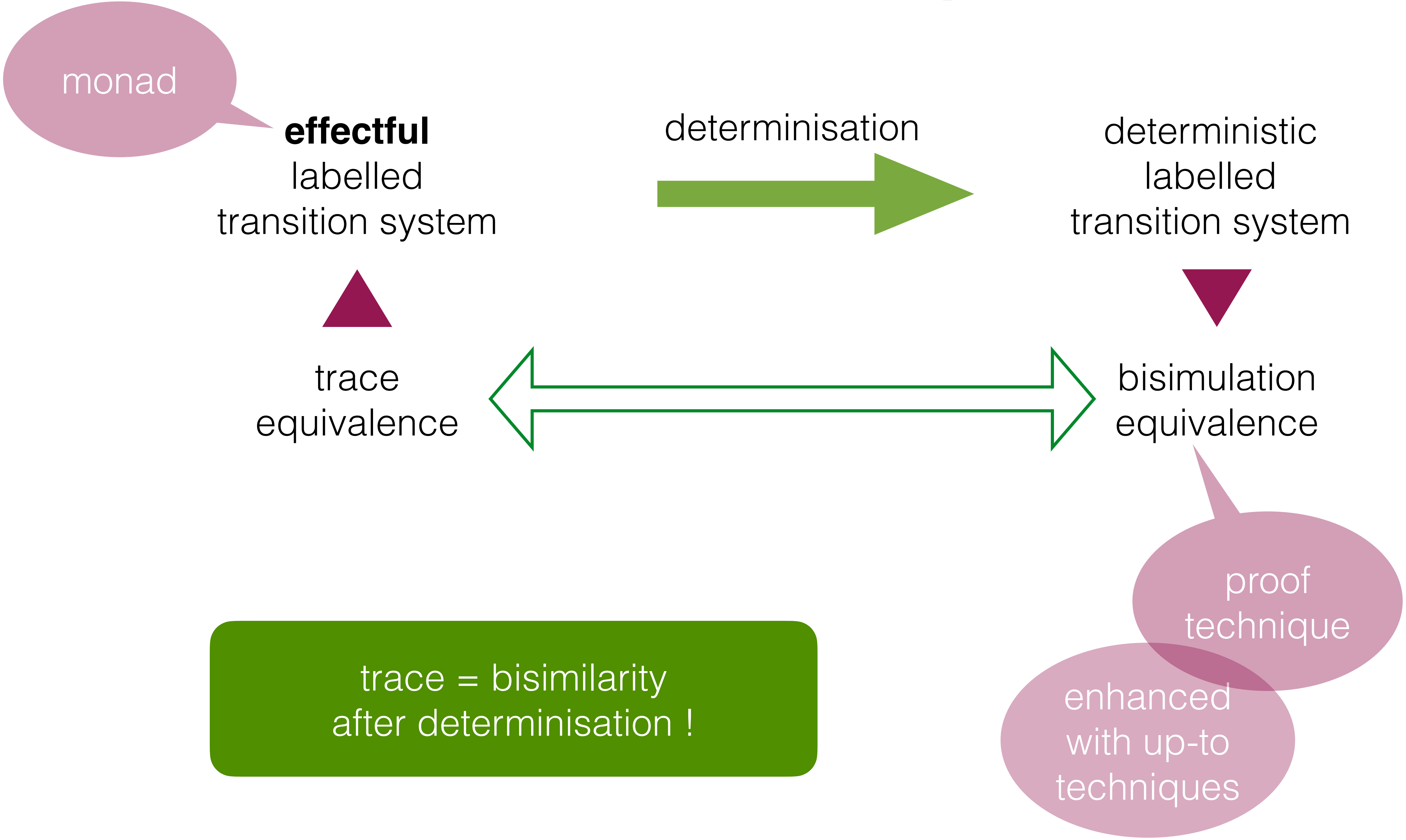
after determinisation !

A proof method for trace equivalence



monad

effectful
labelled
transition system

determinisation

deterministic
labelled
transition system

trace
equivalence

bisimulation
equivalence

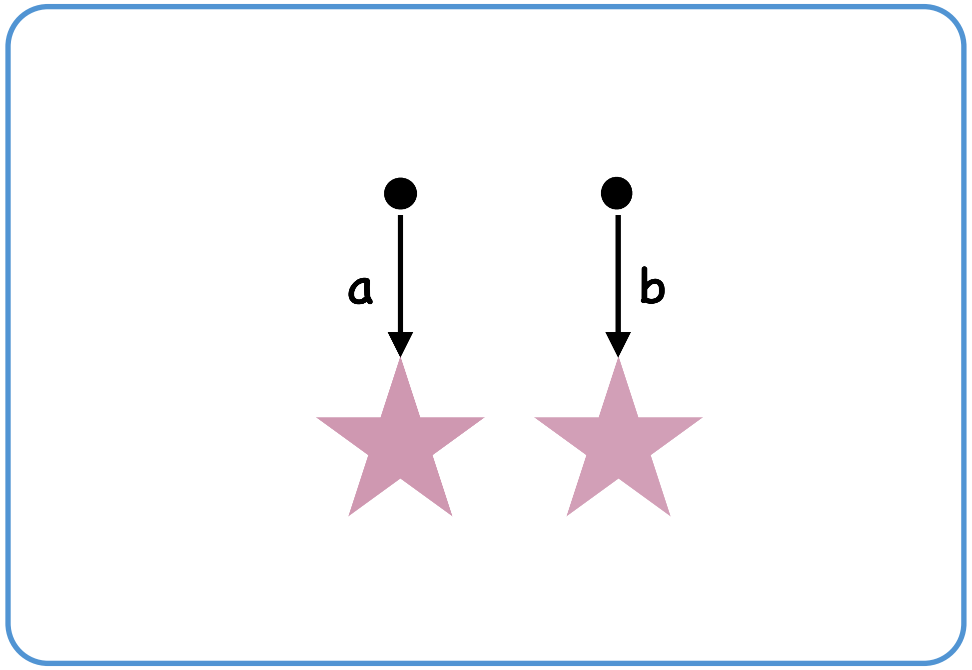
proof
technique

enhanced
with up-to
techniques

trace = bisimilarity
after determinisation !

Effectful Transition Systems

$X \to (MX)^A$



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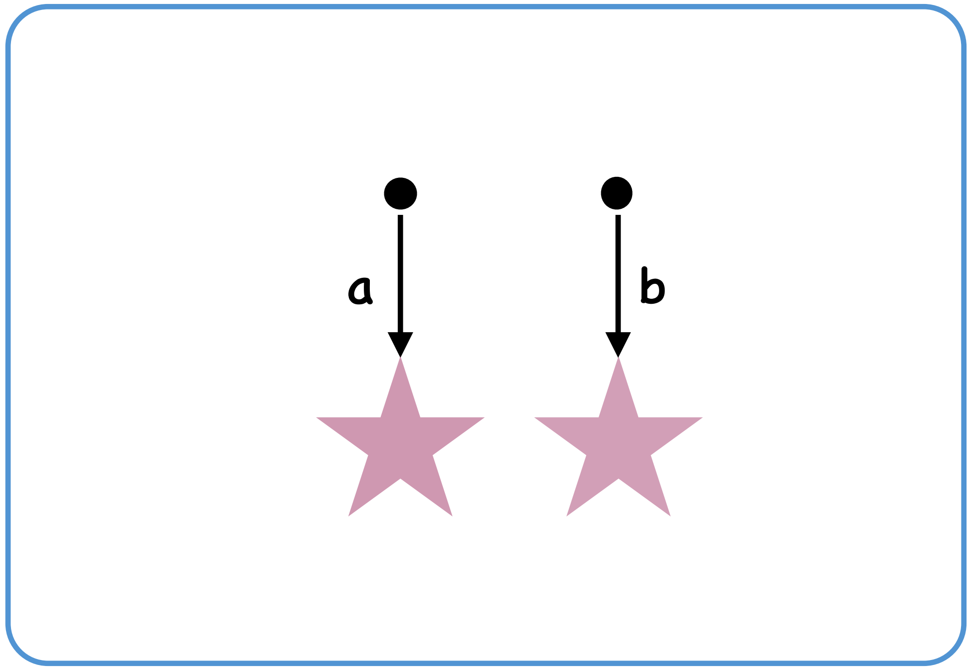


M is a monad, providing algebraic effects

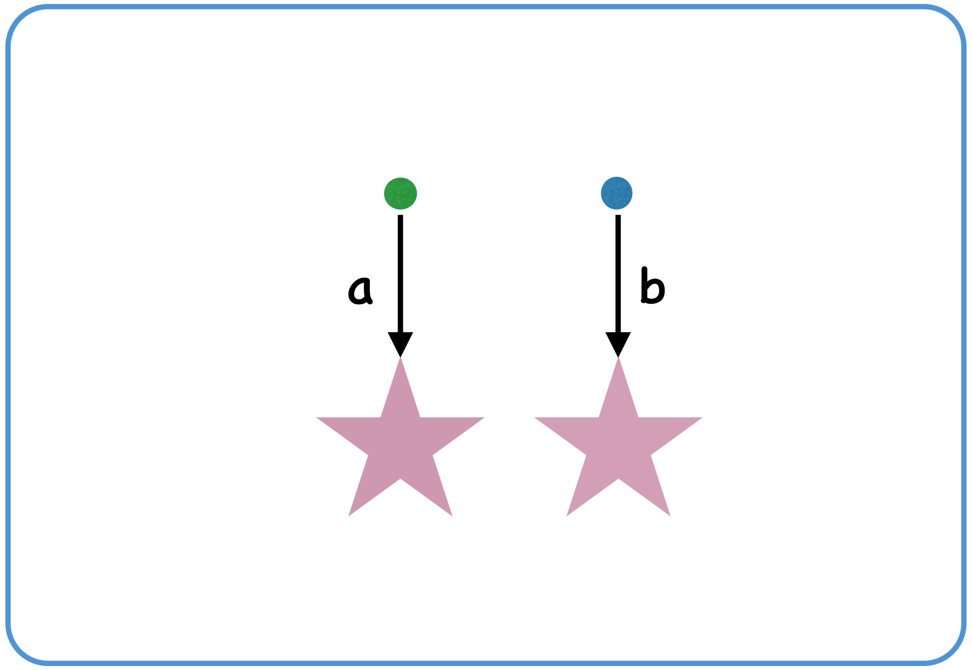
$\mu\colon MM \Rightarrow M \qquad \eta\colon \mathcal{I} \Rightarrow M$

Effectful Trace Semantics

Automata with M-effects

and observations in O

$X \to O \times (MX)^A$



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Effectful Trace Semantics

Automata with M-effects

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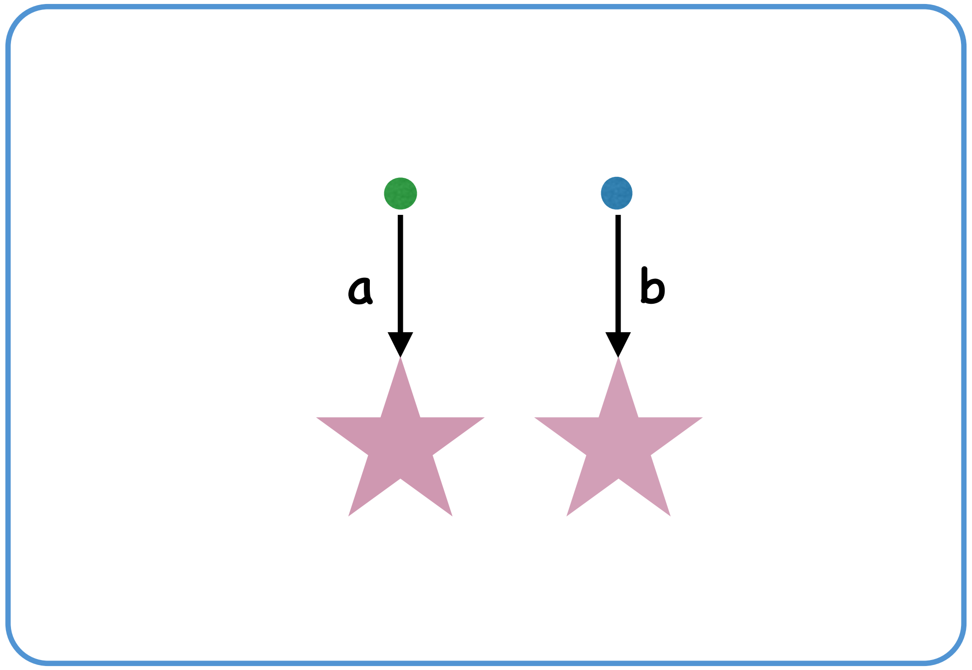
X → O x (MX)^A



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$X \xrightarrow{!} 1 \xrightarrow{\eta} M1$

Simple traces

just a monad!

M-coalgebras

Simple traces

Kurz, Milius, Pattinson, Schröder

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M-coalgebras

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M-coalgebras

$$c\colon X \to MX, \quad c^{\#}\colon MX \xrightarrow{Mc} MMX \xrightarrow{\mu} MX$$

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instance of a determinisation

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Trace given by:

$$\begin{tikzcd}
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Colcombet and Petrisan

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$$\mathrm{trace}\colon X \to (M1)^{\mathbb{N}}$$

Simple traces

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Simple traces

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How about reactive / automata ?

$$\frac{X \to O \times (MX)^A}{X \to O, \quad A \to (X \to MX)}$$

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For each **a** $\in$ **A**, an M-coalgebra

How about reactive / automata ?

$$\frac{X \to O \times (MX)^A}{X \to O, \quad A \to (X \to MX)}$$

For each **a** ∈ A, an M-coalgebra

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$$\frac{X \to O \times (MX)^A}{X \to O, \quad A \to (X \to MX)}$$

For each **a** ∈ A, an M-coalgebra

$$c_a : X \to MX, \quad c_a^{\#} : MX \xrightarrow{Mc_a} MMX \xrightarrow{\mu} MX$$

extends to words
(A* is the free monoid)

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Colcombet and
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$$\mathrm{trace} \colon A^* \to O^X$$

$$\mathrm{trace}(a_1 \ldots a_n) = h \circ c_{a_n}^\# \circ \cdots \circ c_{a_1}^\# \circ \eta$$

- Simple traces suffice for all coalgebra cases
- Everything is an automaton

Thank You

* Simple traces suffice for all coalgebra cases
* Everything is an automaton