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A ZAREMBA-TYPE CRITERION FOR HYPOELLIPTIC DEGENERATE ORNSTEIN–UHLENBECK OPERATORS

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Dedicated to Gisèle Ruiz Goldstein on the occasion of her 60th birthday

Abstract. We prove a cone-type criterion for a boundary point to be regular for the Dirichlet problem related to (possibly) degenerate Ornstein–Uhlenbeck operators in $\mathbb{R}^N$. Our result extends the classical Zaremba cone criterion for the Laplace operator.

1. **Introduction.** We prove a Zaremba-type criterion for a boundary point to be regular for the Dirichlet problem related to degenerate Ornstein–Uhlenbeck operators in $\mathbb{R}^N$ of the type:

$$\mathcal{L} = \operatorname{div}(A\nabla) + \langle Bx, \nabla\rangle . \quad (1)$$

Our result easily follows from a cone-type criterion for the Kolmogorov operators $\mathcal{L} - \partial_t$, recently obtained in [9], and a Tikhonov-type result stated in [12], comparing the $\mathcal{L} - \partial_t$-regularity of the boundary point $z_0 = (x_0, t_0) \in \partial\Omega\times]0, T[$ with the $\mathcal{L}$-regularity of $x_0 \in \partial\Omega$. Here Ω denotes any bounded open subset of $\mathbb{R}^N$, and $]0, T[$ the real interval $\{t \in \mathbb{R} \mid 0 < t < T\}$.

The $N\times N$ matrices $A = (a_{ij})_{i,j=1,\dots,N}$ and $B = (b_{ij})_{i,j=1,\dots,N}$ in (1) are supposed to have real constant coefficients, moreover A is symmetric and nonnegative definite. $x = (x_1, \dots, x_N)$ denotes the point of $\mathbb{R}^N$, while div, ∇ and $\langle\ ,\ \rangle$ stand for the divergence, the Euclidean gradient and the inner product in $\mathbb{R}^N$.

Our crucial assumption is the *hypoellipticity* of the operator (1). We recall that $\mathcal{L}$ is hypoelliptic if every solution of $\mathcal{L}u = f$, in a open subset of $\mathbb{R}^N$, is smooth whenever f is smooth. In [11] some equivalent conditions of hypoellipticity for our operator are listed. One of these conditions is the following one: the linear first order partial differential operators

$$X_i = \sum_{k=1}^{N} a_{ik}\partial_{x_k}, \quad i = 1, \dots, N, \quad \text{and} \quad Y = \langle Bx, \nabla\rangle,$$

satisfy the *Hörmander condition*

$$\text{rank Lie}\{X_1, \dots, X_N, Y\}(x) = N \quad \forall x \in \mathbb{R}^N, \quad (2)$$

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that is, the rank of the Lie algebra generated by $X_1, \dots, X_N, Y$, is maximum at any point of $\mathbb{R}^N$.

Condition (2) is in turn equivalent to the existence of some basis of $\mathbb{R}^N$ such that the matrices A and B take the block form

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}$$

for some $p_0 \times p_0$ symmetric and strictly positive definite matrix A_0, $p_0 \leq N$. Moreover, if $p_0 < N$, i.e., if $\mathcal{L}$ is a degenerate elliptic operator, the matrix B can be written as follows

$$B = \begin{bmatrix} * & * & \dots & * & * \\ B_1 & * & \dots & * & * \\ 0 & B_2 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & B_n & * \end{bmatrix}, \tag{3}$$

where B_j is a $p_{j-1} \times p_j$ matrix with maximum rank p_j; $j = 1, 2, ..., n$, $p_0 \geq p_1 \geq ... \geq p_n \geq 1$ and $p_0 + p_1 + ... + p_n = N$. Every block $*$ is a real constant matrix that does not need to satisfy any particular condition.

Another condition, equivalent to the hypoellipticity of $\mathcal{L}$, that we would like to recall is the *Kalman condition.* More precisely, letting

$$E(s) := \exp\left(-sB\right), \quad s \in \mathbb{R}, \tag{4}$$

the matrix

$$C(t) = \int_0^t E(s) A E^T(s)\, ds, \qquad \text{where } E(s) := \exp\left(-sB\right), \quad s \in \mathbb{R},$$

is strictly positive definite for every $t > 0$.

Over the last few years significant advances have been achieved in the study of degenerate Ornstein–Uhlenbeck operators. We refer the interested reader to the papers [1], [3], [4], [6], [7], [8], and references therein.

Let us now consider the first boundary problem

$$\begin{cases} \mathcal{L}u = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = \varphi. \end{cases}$$

It is quite a standard fact that this problem has a *generalized solution in the sense of Perron–Wiener* for every bounded open set Ω in $\mathbb{R}^N$ and for every $\varphi \in C(\partial\Omega)$ (see, e.g., [2], [5], [10]). If, as usual, H_φ^Ω denotes such a solution, then H_φ^Ω is smooth and satisfies $\mathcal{L}H_\varphi^\Omega = 0$ in Ω in the classical sense. However, in general, it may occur that H_φ^Ω does not assume at the boundary the prescribed datum φ.

A point $x_0 \in \partial\Omega$ is called $\mathcal{L}$-regular for Ω if

$$\lim_{x \to x_0} H_\varphi^\Omega(x) = \varphi(x_0) \quad \forall\, \varphi \in C(\partial\Omega).$$

It is the aim of this note to provide a geometrical $\mathcal{L}$-regularity criterion which is the analogue of the Zaremba cone condition for the Laplacian.

To state our result we have to define the $\mathcal{L}$-cones of $\mathbb{R}^N$. Let us first introduce a group of dilations $(D_r)_{r>0}$, defining

$$\begin{aligned} D_r : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad D_r(x) &= D_r(x^{(p_0)}, x^{(p_1)}, \dots, x^{(p_n)}) \\ &:= (r x^{(p_0)}, r^3 x^{(p_1)}, \dots, r^{2n+1} x^{(p_n)}), \end{aligned}$$

$$x^{(p_i)} \in \mathbb{R}^{p_i}, \quad i = 0, \dots, n, \quad r > 0.$$

Let z_0 be a point of $\mathbb{R}^N$. We call *$\mathcal{L}$-cone with vertex at x_0* a set of the type

$$K := \{D_r(\xi) + E(-r^2T)(x_0) \mid 0 < r < 1, \xi \in V\},$$

where $T > 0$, V is a open ball of $\mathbb{R}^N$, $E(s)$ is the matrix defined in (4).

Our criterion reads as follows.

Theorem 1.1. *Let Ω be a bounded open set in $\mathbb{R}^N$ and let be $x_0 \in \partial\Omega$. Suppose there exists an $\mathcal{L}$-cone K with vertex at x_0 such that*

$$K \subseteq \mathbb{R}^N \backslash \Omega. \tag{5}$$

Then x_0 is $\mathcal{L}$-regular for Ω.

Proof. Let $T > 0$ and $V \subseteq \mathbb{R}^N$ be a suitable Euclidean ball such that

$$K = \{D_r(\xi) + E(-r^2T)(x_0) \mid 0 < r < 1, \xi \in V\}$$

is contained in $\mathbb{R}^N \backslash \Omega$. Let us consider now the evolution operator

$$\mathcal{H} := \mathcal{L} - \partial_t$$

and define the cylindrical domain in $\mathbb{R}^{N+1}$

$$\mathcal{O} := \Omega \times]0, 1[.$$

As pointed out in [9], for every $\phi \in C(\partial\mathcal{O})$, the boundary value problem

$$\begin{cases} \mathcal{H}u = 0 \text{ in } \mathcal{O}, \\ u|_{\partial\mathcal{O}} = \phi, \end{cases}$$

has a generalized solution in the sense of Perron–Wiener. By the Tikhonov-type result stated in [12, Teorema 2], the point $x_0 \in \partial\Omega$ is $\mathcal{L}$-regular for Ω if and only if the point $z_0 = (x_0, t_0), t_0 \in]0, 1[$, is $\mathcal{H}$-regular for $\mathcal{O}$.

On the other hand, by Theorem 6.2 in [9], z_0 is $\mathcal{H}$-regular if the $\mathcal{H}$-cone with vertex at z_0

$$\hat{K} = \{(D_r(\xi) + E(-r^2T)(x_0), t_0 - r^2T) \mid 0 < r < 1, \xi \in V\}$$

is contained in $\mathbb{R}^{N+1} \backslash \mathcal{O}$.

This condition is satisfied thanks to the inclusion (5). □

Remark 1. If $A = \mathbb{I}_N$ is the identity matrix in $\mathbb{R}^N$ and $B = 0$, then $\mathcal{L}$ in (1) becomes the Laplace operator and Theorem 1.1 recovers the classical Zaremba cone criterion [13].

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