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Liouville-type theorems for Kolmogorov and Ornstein–Uhlenbeck operators

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Abstract. We collect Liouville-type properties that hold true for Kolmogorov operators with constant coefficients and for their time-stationary counterpart, the Ornstein–Uhlenbeck operators. In particular, we discuss uniqueness results for solutions and sub-solutions in L^p-spaces, for solutions in the whole space or in halfspaces bounded just from one-side. Polynomial Liouville properties and a Liouville theorem "at $t = -\infty$" are also presented.

1. INTRODUCTION

We consider Kolmogorov operators in $\mathbb{R}^{N+1}$ of the following type

$$\mathcal{L} = \operatorname{div}(A\nabla) + \langle Bx, \nabla\rangle - \partial_t \ , \tag{1.1}$$

and their corresponding stationary counterparts in $\mathbb{R}^N$, the degenerate Ornstein–Uhlenbeck operators

$$\mathcal{L}_0 = \operatorname{div}(A\nabla) + \langle Bx, \nabla\rangle \ . \tag{1.2}$$

The point $x = (x_1, \cdots, x_N)$ belongs to $\mathbb{R}^N$, t to $\mathbb{R}$, and div, ∇, $\langle\cdot,\cdot\rangle$ stand respectively for the divergence, the Euclidean gradient and the inner product in $\mathbb{R}^N$. $A = (a_{ij})_{i,j=1,\dots,N}$ and $B = (b_{ij})_{i,j=1,\dots,N}$ are $N \times N$ matrices with real constant coefficients, A is symmetric and non-negative definite. We will suppose that A and B satisfy additional suitable conditions, that we are going to describe in the sequel, in order that the operator $\mathcal{L}$, and, as a consequence, $\mathcal{L}_0$, be hypoelliptic.

We recall that an operator $\mathcal{L}$ is hypoelliptic if every distributional solution of $\mathcal{L}u = f$, in a open subset of $\mathbb{R}^{N+1}$, is of class C^∞ whenever f is of class C^∞.

We know (see [12]) that the following conditions on the operator $\mathcal{L}$ in (1.1) are equivalent:

(i) The operator $\mathcal{L}$ is hypoelliptic.

(ii) There exist bases of $\mathbb{R}^N$ such that the matrix A takes the following block form:

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{1.3}$$

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for some $p_0 \times p_0$ symmetric and strictly positive definite matrix A_0, $p_0 \leq N$. Moreover, if $p_0 < N$, i.e., if $\mathcal{L}$ is a degenerate elliptic-parabolic operator, the matrix B can be written as follows

$$B = \begin{bmatrix} * & * & \cdots & * & * \\ B_1 & * & * & * & * \\ 0 & B_2 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_n & * \end{bmatrix} , \tag{1.4}$$

where B_j is a $p_j \times p_{j-1}$ matrix with maximum rank p_j; $j = 1, 2, \cdots, n$, $p_0 \geq p_1 \geq \cdots \geq p_n \geq 1$ and $p_0 + p_1 + \cdots + p_n = N$. Every block $*$ is a real constant matrix that does not need to satisfy any particular condition.

(iii) The first order differential operators

$$X_i = \sum_{k=1}^{N} a_{ik} \partial_{x_k} \quad , \quad i = 1, \cdots, N \quad \text{and} \quad Y = \langle Bx, \nabla \rangle - \partial_t \ , \tag{1.5}$$

satisfy the *Hörmander condition*

$$\operatorname{rank} \operatorname{Lie}\{X_1, \cdots, X_N, Y\}(z) = N + 1 \qquad \forall z \in \mathbb{R}^{N+1} \ , \tag{1.6}$$

that is, the rank of the Lie algebra generated by $X_1, \cdots, X_N, Y$ is maximum at any point of $\mathbb{R}^{N+1}$.

(iv) Letting

$$E(s) := e^{-sB} \quad , \quad s \in \mathbb{R} \ , \tag{1.7}$$

the matrix

$$C(t) = \int_0^t E(s) A E^T(s) \, ds$$

satisfies the *Kalman condition*, that is, C is strictly positive definite for every $t > 0$.

Thus, we suppose the matrices A and B to be of the form (1.3) and (1.4). Lanconelli and Polidoro in [12] proved that the operator $\mathcal{L}$ in (1.1) is left translation invariant with respect to the Lie group $\mathbb{K} = (\mathbb{R}^{N+1}, \cdot)$ with composition law

$$(x, t) \cdot (x', t') = (x' + E(t')x, t + t') \ ,$$

where E represents the matrix defined in (1.7).

Furthermore in the particular case that every block $*$ of the matrix B in (1.4) is the zero matrix of suitable dimensions, the operator $\mathcal{L}_0$ is homogeneous of degree two with respect to the group of dilations

$$\begin{aligned} D_r : \mathbb{R}^N \to \mathbb{R}^N \quad , \quad D_r(x) &= D_r(x^{(p_0)}, x^{(p_1)}, \cdots, x^{(p_n)}) \\ &:= (r x^{(p_0)}, r^3 x^{(p_1)}, \cdots, r^{2n+1} x^{(p_n)}) \ , \end{aligned}$$

and $\mathcal{L}$ is homogeneous of degree two with respect to the group of dilations

$$\delta_r : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1} \quad , \quad \delta_r(x, t) := (D_r(x), r^2 t) \ ; \tag{1.8}$$

where $x^{(p_i)} \in \mathbb{R}^{p_i}$, $i = 0, \cdots, n$, and $r > 0$.

We denote by $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}_0$ the operators $\mathcal{L}$ and $\mathcal{L}_0$ whenever they satisfy the additional hypothesis of homogeneity.

As the dimension of the vectorial space generated by the vector fields $X_i(x), i = 1, \cdots, N$ in (1.5) and $\langle Bx, \nabla\rangle$ is not constant in x, there does not exist a composition law making $\mathcal{L}_0$ left translation invariant (see [3, Proposition 1.2.13]).

We conclude the introduction with a celebrated example. Let $\mathbb{I}_n$ be the identity $n \times n$ matrix and let $N = 2n$. Suppose that the matrices A and B are of the type:

$$A = \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ \mathbb{I}_n & 0 \end{pmatrix} .$$

The prototype of the operators introduced in 1934 by Kolmogorov [11] in studying diffusion phenomena from a probabilistic point of view is

$$\mathcal{K} = \operatorname{div}(A\nabla) + \langle Bx, \nabla\rangle - \partial_t = \sum_{i=1}^{n} \partial_{x_i}^2 + \sum_{i=1}^{n} x_i \partial_{x_{n+i}} - \partial_t \ \text{ in } \mathbb{R}^{2n+1} .$$

In the model of Kolmogorov the positive solutions of $\mathcal{K}u = 0$ are probability densities of a system having $2n$ *degrees of freedom*. The $2n$ dimensional space is the phase space: $(x_1, \cdots, x_n)$ is the velocity vector and $(x_{n+1}, \cdots, x_{2n})$ is the vector of the positions of the system.

We note that in this case:

$$C(t) = \int_0^t e^{-sB}\, A\, e^{sB^T}\, ds = \begin{pmatrix} t\,\mathbb{I}_n & -\dfrac{t^2}{2}\,\mathbb{I}_n \\ -\dfrac{t^2}{2}\,\mathbb{I}_n & \dfrac{t^3}{3}\,\mathbb{I}_n \end{pmatrix} > 0 \quad , \quad \forall\, t > 0 .$$

This paper is organised as follows. In section 1, we collect Liouville-type results in L^p, $p \in]1, \infty)$, for solutions and sub-solutions to $\mathcal{L}u = 0$, together with an application to the Cauchy problem. Section 2 is devoted to polynomial Liouville properties and a Liouville theorem at $t = -\infty$ for the Kolmogorov operators satisfying the homogeneity assumption. In section 3 we discuss the case L^∞, that is the Liouville theorem for bounded solutions, both for $\mathcal{L}$ and its “stationary-time” $\mathcal{L}_0$ counterpart. In section 4, one-side Liouville theorems for $\mathcal{L}_0$ are derived from the previous results.

For the sake of brevity, we do not provide here the proofs of the presented results, but instead refer the interested reader to the corresponding publications. In those publications there can also be found related citations to additional bibliography. We also mention the recent survey [1] by Anceschi and Polidoro on the classical theory for Kolmogorov operators.

Remark 1.1 (sub-Kolmogorov operators)**.** We observe that under suitable conditions on the matrix B (see [6]), we may replace the second order part div $(A\nabla)$ in the operator (1.1) by a sub-Laplacian on a Carnot group $\mathbb{G}$ in $\mathbb{R}^N$. In this case, we obtain the following class of evolution operators left invariant and homogenous on a Lie group in $\mathbb{R}^{N+1}$ (that we construct following the *link* procedure introduced in [6]):

$$\mathcal{L}_{\mathbb{G}} := \mathcal{L}_{\mathbb{G}0} - \partial_t = \mathit{\Delta}_{\mathbb{G}} + \langle Bx, \nabla\rangle - \partial_t \text{in } \mathbb{R}^{N+1} .$$

An example of an operator belonging to this class is

$$\Delta_{\mathbb{H}_1} + x_1\partial_{x_4} - \partial_t := (\partial_{x_1} + 2x_2\partial_{x_3})^2 + (\partial_{x_2} - 2x_1\partial_{x_3})^2 + x_1\partial_{x_4} - \partial_t \text{in } \mathbb{R}^5 ,$$

where $\mathbb{H}_1$ is the Heisenberg group in $\mathbb{R}^3$.

All the presented results for $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}_0$, hold true also for $\mathcal{L}_{\mathbb{G}}$ and $\mathcal{L}_{\mathbb{G}0}$, respectively.

2. L^p-LIOUVILLE THEOREMS

We present some L^p-Liouville theorems that, in a *suitable form*, hold true for the solution and sub-solutions to $\mathcal{L}u = 0$ (see [8, 2]).

We observe that in the case of harmonic functions, the L^p-Liouville theorem is an easy consequence of the Gauss mean value property. Indeed, let $u \in L^p(\mathbb{R}^N)$, $1 \leq p < \infty$, and $\Delta u = 0$ in $\mathbb{R}^N$ (we denote by μ the Lebesgue measure).

Then,

$$|u(x)| = \frac{1}{\mu(B_r(x))} \left| \int_{B_r(x)} u(y)\, dy \right|$$

$$\leq \left(\frac{1}{\mu(B_r(x))} \right)^{1/p} \|u\|_{L^p(\mathbb{R}^N)} \to 0 \quad \text{as } r \to \infty \quad , \quad \forall\, x \in \mathbb{R}^N .$$

However this argument does *not* work for the heat equation because the kernel in the Pini-Watson mean value Theorem for caloric functions is *unbounded*.

The first result is an L^p-Liouville theorem where we do not require any sign restriction on the solution u. The *weight* $e^{t\,\mathrm{tr}(B)}$ represents a right-invariant measure on $\mathbb{K}$ (see [2]), where by $\mathrm{tr}(B)$ we denote the trace of the matrix B. We recall that in the particular case where the operator is homogenous, we have that $\mathrm{tr}(B) = 0$.

Theorem 2.1. *Let u be a solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{N+1}$. If*

$$\int_{\mathbb{R}^{N+1}} |u(x,t)|^p\, e^{t\,\mathrm{tr}(B)}\, dxdt < \infty$$

for some $p \in [1, \infty)$, then $u \equiv 0$.

For $p = \infty$ the validity of a similar result depends on the real part of the eigenvalues of the matrix B. We will examine the case $\sup_{\mathbb{R}}^{N+1} |u| < \infty$ in section 4. Now, we wish to present L^p-Liouville properties for the sub-solutions to $\mathcal{L}$.

Theorem 2.2. *Let $u \in C^2(\mathbb{R}^{N+1})$ be a non-negative solution to $\mathcal{L}u \geq 0$ in $\mathbb{R}^{N+1}$. If*

$$\int_{\mathbb{R}^{N+1}} |u(x,t)|^p\, e^{t\,\mathrm{tr}(B)}\, dxdt < \infty$$

for some $p \in [1, \infty)$, then $u \equiv 0$.

Actually, for the sub-solutions the result is more precise:

Theorem 2.3. *Let $u \in C^2(\mathbb{R}^{N+1})$ be a solution to $\mathcal{L}u \geq 0$ in $\mathbb{R}^{N+1}$. If*

$$\int_{\mathbb{R}^{N+1}} |u(x,t)|^p\, e^{t\,\mathrm{tr}(B)}\, dxdt < \infty$$

for some $p \in [1, \infty)$, then $u \leq 0$ in $\mathbb{R}^{N+1}$.

The result becomes sharp when the operator is homogeneous. Let $Q := p_0 + 3p_1 + \cdots + (2n+1)p_n + 2$. The number Q denotes the *homogeneous dimension* of $\mathbb{K}$ with respect to the group of dilations $(\delta_r)_{r>0}$ defined in (1.8). We can state the following theorem.

Theorem 2.4. *Let* $u \in L^1_{loc}$ *be a solution to* $\widetilde{\mathcal{L}}u \geq 0$ *in* $\mathbb{R}^{N+1}$*, in the sense of distributions. If there exists* $p \in [1, 1 + 2/(Q-2)]$ *such that*

$$\int_{\mathbb{R}^{N+1}} |u(x,t)|^p \, dxdt < \infty \ ,$$

then

$$u \equiv 0 \qquad a.e. \ in \ \mathbb{R}^{N+1} \ .$$

Moreover, for every $p > 1 + 2/(Q-2)$*, there exists* $u \in L^p(\mathbb{R}^{N+1}), u \leq 0$*,* $u \not\equiv 0$*, such that*

$$\widetilde{\mathcal{L}}u \geq 0 \ in \ \mathbb{R}^{N+1}, \ in \ the \ sense \ of \ distributions \ .$$

We complete this section with an application to the Cauchy problem: the following Tikhonov-type theorem for all the hypoelliptic operators $\mathcal{L}$ in (1.1) holds.

Corollary 2.5. *Let us denote by* Ω *the half-space* $\{(x,t) \in \mathbb{R}^{N+1} \, : \, t > 0\}$*. Any classical solution* $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ *to the Cauchy problem*

$$\begin{cases} \mathcal{L}u = 0 & in \ \Omega \\ u(x,t) = 0 & for \ t = 0 \end{cases}$$

is identically zero on Ω *if*

$$\int_0^\infty \int_{\mathbb{R}^N} |u(x,t)|^p \, e^{t \operatorname{tr}(B)} \, dxdt < \infty \ ,$$

for some $p \in [1, \infty)$*.*

3. Asymptotic Liouville theorems

We cannot expect a non-negative solution to $\widetilde{\mathcal{L}}u = 0$ in $\mathbb{R}^{N+1}$ to be constant without adding any extra condition. This can be easily seen, e.g., by considering the following function

$$u(x,t) = e^{x_1 + \cdots + x_N + Nt} \ ,$$

which is a strictly positive non constant solution in $\mathbb{R}^{N+1}$ to the heat equation

$$\sum_{i=1}^N \partial^2_{x_i} u - \partial_t u = 0 \ .$$

However, if we assume a polynomial growth on u as $|x|$ or t goes to ∞, we can recover the following Liouville-type theorems (see [5, Theorem 2], [7, Theorem 2.3]).

Theorem 3.1. *Let* $u \in C^\infty(\mathbb{R}^N \times]-\infty, 0)) \cap C(\mathbb{R}^N \times]-\infty, 0])$ *be a non-negative solution to* $\widetilde{\mathcal{L}}u = 0$ *in* $\mathbb{R}^N \times]-\infty, 0)$*.*
If there exists a positive $m \in \mathbb{R}$ *such that*

$$u(x,0) = O(|x|^m) \quad as \ |x| \to \infty \ ,$$

then u *is constant.*

Theorem 3.2. *Let* u *be a non-negative solution to* $\widetilde{\mathcal{L}}u = 0$ *in* $\mathbb{R}^{N+1}$*.*
If there exists a positive $m \in \mathbb{R}$ *such that*

$$u(0,t) = O(t^m) \quad as \ t \to \infty \ ,$$

then u *is constant.*

In the particular case that B is the zero matrix, the asymptotic estimate $u(0,t) = O(t^m)$ can be replaced by

$$u(0,t) = O(e^{\varepsilon t}) \quad \text{as } t \to \infty \ ,$$

for every $\varepsilon > 0$ (see [9, Theorem 5.1]).

The previous result is a particular case of the following *polynomial Liouville theorem.*

Theorem 3.3. *Let* $u : \mathbb{R}^{N+1} \to \mathbb{R}$ *be such that*

$$\begin{cases} \widetilde{\mathcal{L}} u = p & \text{in } \ \mathbb{R}^{N+1} \ , \\ u \geq q & \text{in } \ \mathbb{R}^{N+1} \ , \end{cases}$$

where p *and* q *are polynomial functions. Assume*

$$u(0,t) = O(t^m) \qquad \text{as } t \to \infty \ , \tag{3.1}$$

for a suitable $m > 0$*. Then,* u *is a polynomial function.*

Without adding any extra condition on the non-negative solution u to our equation $\widetilde{\mathcal{L}} u = 0$, as already observed, u is not necessarily constant. Nevertheless, it has been proved the "u is constant at $t = -\infty$" (see [5, Theorem 1], [7, Theorem 3.1]), in the following sense.

Theorem 3.4. *Let* u *be a non-negative solution to* $\widetilde{\mathcal{L}} u = 0$ *in the half-space*

$$S_T = \mathbb{R}^N \times] - \infty, T[\quad , \quad T \in \mathbb{R} \ .$$

Then $\lim_{t \to -\infty} u(x,t)$ *exists for every* $x \in \mathbb{R}^N$ *and it is independent of* x*.*

More precisely,

$$\lim_{t \to -\infty} u(x,t) = \inf_{S_T} u \qquad \text{for every } \ x \in \mathbb{R}^N \ .$$

4. L^∞-LIOUVILLE THEOREMS

If we suppose that the absolute value of the solution u to $\widetilde{\mathcal{L}} u = 0$ is bounded, the Liouville theorem for $\widetilde{\mathcal{L}}$ follows like a simple corollary of the results of the previous section.

Theorem 4.1. *Let* u *be a bounded solution to* $\widetilde{\mathcal{L}} u = 0$ *in* $\mathbb{R}^{N+1}$*. Then,* u *is constant.*

We observe that if the operator $\widetilde{\mathcal{L}}$ satisfies the L^∞-Liouville property (i.e. any bounded solution to the equation $\widetilde{\mathcal{L}} u = 0$ in $\mathbb{R}^{N+1}$ is constant) also $\widetilde{\mathcal{L}}_0$ has the same property for the solution v to $\widetilde{\mathcal{L}}_0 v = 0$ in $\mathbb{R}^N$.

Actually this last result is a particular case of a theorem due to Priola and Zabczyk [15, Theorem 3.1], that holds under more general hypotheses on the matrix B (that include the homogeneity). More precisely, the Priola and Zabczyk theorem for the operator $\mathcal{L}_0$ in (1.2), takes this form.

Theorem 4.2. *Any bounded solution* v *to* $\mathcal{L}_0 v = 0$ *in* $\mathbb{R}^N$ *is constant*

$$\Longleftrightarrow$$

the real part of every eigenvalue of the matrix B *is non-positive .*

5. One-side Liouville theorems

As a consequence of the asymptotic Liouville theorems for the Kolmogorov operators recalled in section 3, we obtain for their stationary-time counterparts (the Ornstein–Uhlenbeck operators), Liouville-type properties without requiring any *a priori* asymptotic behaviour for the solutions.

We remark that our class of operators $\widetilde{\mathcal{L}}_0$ also contains "parabolic" type operators like, e.g. the following "forward-backward" heat operator

$$\mathcal{L}_0 := \ \partial^2_{x_1} + x_1\partial_{x_2} \qquad \text{in } \ \mathbb{R}^2 \ . \tag{5.1}$$

We start by presenting a polynomial one-side Liouville theorem.

Theorem 5.1. *Let* $v : \mathbb{R}^N \to \mathbb{R}$ *be such that*

$$\begin{cases} \widetilde{\mathcal{L}}_0 v = p & \text{in } \ \mathbb{R}^N \ , \\ v \geq q & \text{in } \ \mathbb{R}^N \ , \end{cases}$$

where p *and* q *are polynomial functions. Then,* v *is a polynomial function.*

As a particular case, we have for our class of Ornstein–Uhlenbeck operators, a Liouville theorem analogous to the one for the classical Laplace operator.

Theorem 5.2. *Let* $v : \mathbb{R}^N \to \mathbb{R}$ *be such that*

$$\begin{cases} \widetilde{\mathcal{L}}_0 v = 0 & \text{in } \ \mathbb{R}^N \ , \\ v \geq 0 & \text{in } \ \mathbb{R}^N \ , \end{cases}$$

Then, v *is constant.*

This result follows very easily from Theorem 3.2. Indeed, let us define $u(x,t) = v(x)$. Then u is a non-negative solution to the equation $\widetilde{\mathcal{L}}u = 0$ in $\mathbb{R}^{N+1}$. Moreover, $u(0,t) = v(0) = O(1)$ as $t \to \infty$. Then, u is constant, and so v is constant. In a similar way, one obtains the polynomial Liouville theorem for $\widetilde{\mathcal{L}}_0$ from Theorem 3.3.

From the Priola and Zabczyk theorem [15, Theorem 3.1], it is clear that we cannot extend Theorem 5.2 to the class of operators $\mathcal{L}$ when B has at a least a positive eigenvalue. It is an open problem if in the general case, where all the eigenvalues of the matrix B have non positive real parts, the solutions to the equation $\mathcal{L}_0 v = 0$ bounded *just from one side* have to be constant.

Very recently, with Priola and Lanconelli [10], we obtained the one-side Liouville property for operators of the type

$$\sum_{i=1}^{N} \partial^2_{x_i} u + \langle Bx, \nabla \rangle \ ,$$

when B is diagonalizable over the complex field with all its eigenvalues on the imaginary axis.

6. A brief historical note

Actually, Augustin-Louis Cauchy was the first to announce and prove, on the 23rd of December 1844, the earliest statement of the theorem nowadays known as the "the Liouville theorem",

any bounded entire function of a single complex variable has to be constant,

two weeks after Joseph Liouville, on the 9th of December, announced the result in the special case of doubly periodic functions,

a doubly-periodic holomorphic function has to be constant.

Both the results were published in the Comptes Rendus de l'Académie des Sciences, Paris [13, 4]. For an historical account concerning the priority issue we refer to the chapter *The Discovery of Liouville's Theorem* of [14] and the historical note *Cauchy and Liouville, a question of priority* in [16].

Lützen asserts that "Liouville justly deserves the honor of having his name attached to the theorem for the following three reasons. 1° Liouville was the first to publish the theorem in the doubly periodic case, Cauchy's 1843 theorems being clearly different from, although closely related to, Liouville's theorem. 2° Liouville was the first to discover the fundamental importance of the theorem. 3° Liouville probably had arrived at the general form of Liouville's theorem before Cauchy". Contrarily, Serrin and Zou, in their historical note in [16], argue that "Lützen's presentation is partly marred by championship of Liouville's priority claims".

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